Classical Control Systems

Lecture Notes
for
Undergraduate
Course.

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Classical Control Systems

Based on the lecture contents of the "BCS Course"
a HZ–SMU collaboration

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Preface

This hand-witten lecture note in classical control systems covers the course contents I teach in Shanghai Maritime University. The course aims at providing a comprehensive yet rigorous introduction to classical control theory and some widely used analysis & design tools used in engineering practice.

Although there are a few pages of knowledge recap, introductory level of calculus, physics, and complex analysis knowledge is expected from the reader.

My course contains roughly 10x90min lectures:

Lecture 1 & 2 : chapter 1, 2, 3

Lecture 3 : chapter 8, 10

Lecture 4 : chapeer 4

Lecture 5 : chapter 5, 9

Lecture 6 : chapter 7

Lecture 7 & 8 & 9 : chapter 6, 9

Lecture 10 : review

Table of Contents

Introduction

Control Systems & key concepts.

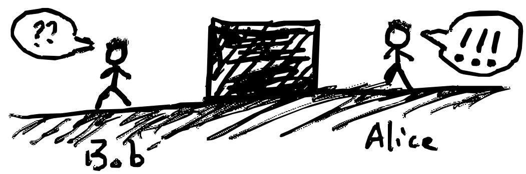
→ What is control?

Engineering the target system to obtain the desired output.

→ Systems, hmmmm..... what can be a system?

Almost everything! Depends on your interest.

A large piece of metal on the ground:



Bob wants to travel to Japan, obviously this large metal piece is not so interesting for him. As long as it does not block his way, for him, that's not a system.

Alice wants to make a heat exchanger! This is such an interesting system for her because metals typically exchange heat quite well! She may test it right now!

→ What kind of systems are we interested in?

"Causal LTI SISO systems"

Causal LTI SISO Systems

decomposition:

Causal: output only depends on the past & present input, not the future.

The system should remain stationary before a reference time t_0.

$\forall\ t_1 \neq t_2$, $t_1, t_2 < t_0$, we have $x(t_1) = x(t_2)$, $y(t_1) = y(t_2)$

Typically we let $t_0 = 0$.

LTI: Linear Time-Invariant

Linear:

- Homogeneity:

  $x(t) \mapsto y(t)$

  then, $\alpha x(t) \mapsto \alpha y(t)$

- Additive

  $x_1 \mapsto y_1$

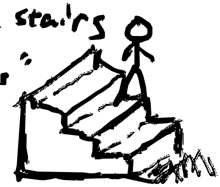
  $x_2 \mapsto y_2$

  then $x_1 + x_2 \mapsto y_1 + y_2$

Time-Invariant: $x(t) \mapsto y(t)$, then $x(t+a) \mapsto y(t+a)$

Example:

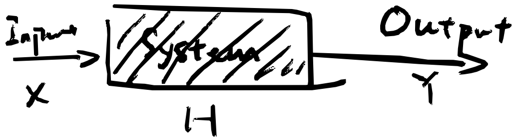
you going down the stairs



The "going-down-stairs" process does not matter if you go at 10:30 or 12:30.
(what might matter is if there's food left in the canteen )

SISO: Single Input, Single Output.

Hmmmm ....

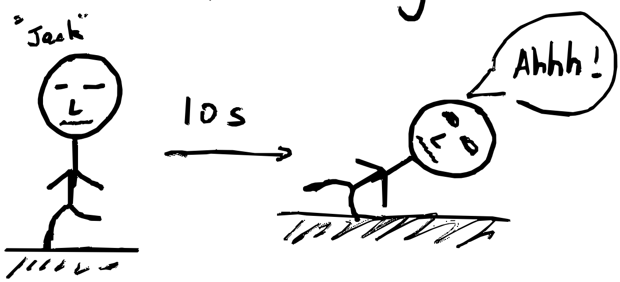


Everything seems so simple, too simple with only

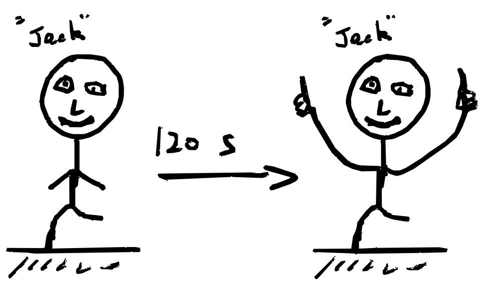
$$Y = HX$$

Say Jack have to stand-up on one foot!

Jack wants himself to keep balance, he needs to control his body.



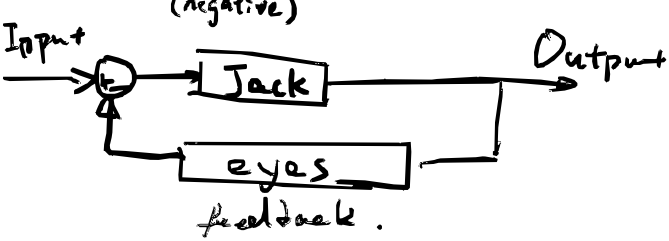
Jack first tried with his eyes closed, but he falls after 10 seconds.



Jack now opens his eyes. His eyes give him feedback for his body position.

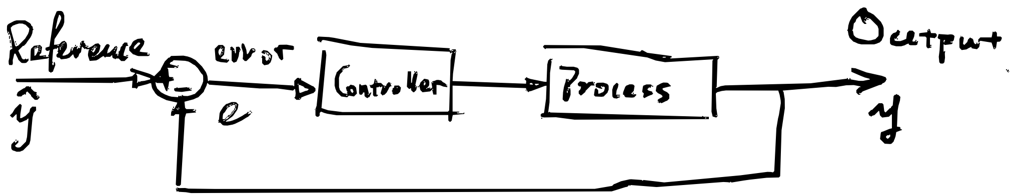
Now Jack has kept his balance for 120 seconds! Yayyyyy!

Add feedback!
(negative)

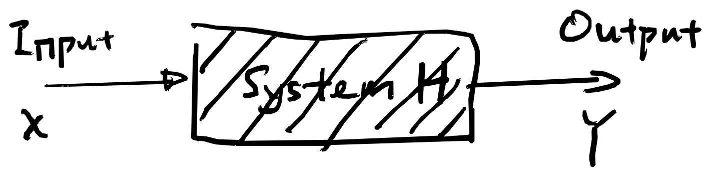


Control loops

"A control loop with negative unit feedback"

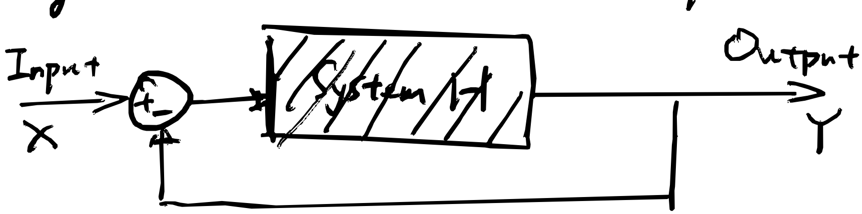


In general, for a given unknown (process) system, we have



Transfer function: $H = \frac{Y}{X}$
(Open-loop)

Closing the system with a controll loop:



$$(X - Y)H = Y$$
$$XH = (H+1)\,Y$$

$$\Rightarrow \frac{Y}{X} = \frac{H}{1+H}$$ transfer function (closed loop)

Feedback : pros & cons

Pros:

- deal with dynamics
- robustness
- modular
- gather more information

Cons:

- complexity ↑
- could bring unstable behavior
- amplify noise.

Stability ! (quite important)

The Math:

given a system $h(t)$, stability of $h(t)$ requires:

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

if causal then $\int_0^{\infty} |h(t)|\,dt < \infty$

Absolutely integrable impulse response.

But to be simple, we need:

$$\text{finite input} \Longrightarrow \text{System} \Longrightarrow \text{finite output.}$$

"Bounded-Input-Bounded-Output (BIBO) stability"

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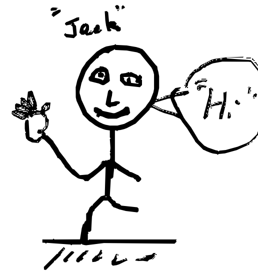
→ What does/might stability mean?
(finite integral)

Physical Meaning!

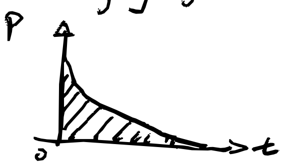


Back to our balancing guy Jack!

Now here's some bad guy pushing jack.

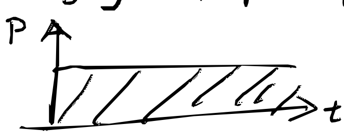
The power of the bad guy is P

① Bad guy gets tired over time



"Jack" is still there

② Bad guy keeps pushing



"Jack" falls

③ Powerful bad guy!

"Jack" might get injured.

The shaded area: $\int_0^\infty P\,dt$: the energy Jack consumes to balance himself!

▲ Infinite integral ($h(t)$ diverge) $\Rightarrow$ infinite energy $\Rightarrow$ System break! (unstable)

Knowledge Recap

Pt. A | Complex Numbers "z"

$$z = a + jb \dashrightarrow \begin{array}{l} a: Re(z) \text{ "Real part"} \\ b: Im(z) \text{ "Imaginary part"} \end{array}$$

$$j^2 = -1$$

Complex numbers z are two dimensional

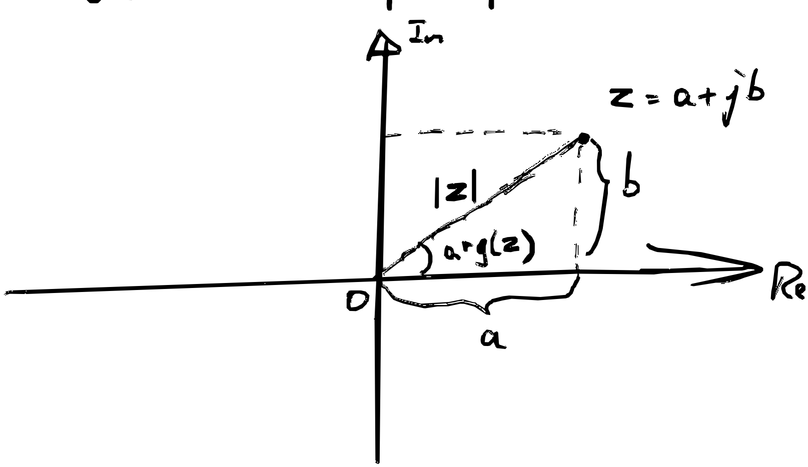
We have the Modulus of a complex number z

$$|z| = \sqrt{Re^2(z) + Im^2(z)} = \sqrt{a^2 + b^2}$$

Naturally, the argument of z arises:

$$\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$$

Visualization in complex plane



8 # Euler's Formula

$\forall \theta \in \mathbb{R}, \; e^{j\theta} = \cos\theta + j\sin\theta$

Thus we also obtain the Euler's identity:

$$e^{j\pi} + 1 = 0$$

Observe $|e^{j\theta}| = |\cos\theta + j\sin\theta|$

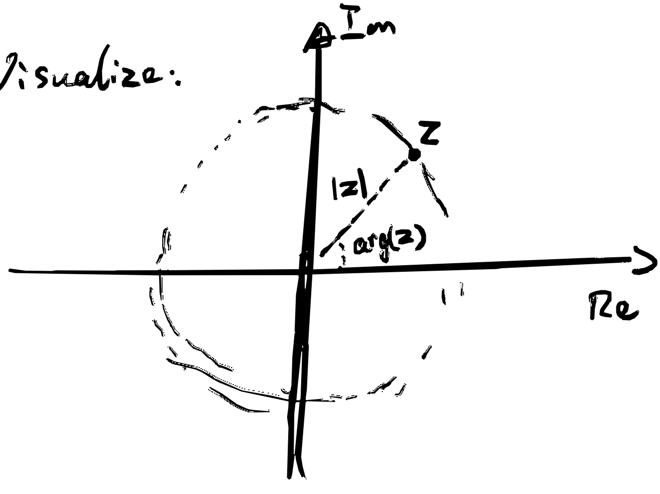
$$= \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1$$

We are ready to present the polar form of any complex number z:

$$z = |z| e^{j\arg(z)}$$

Visualize:

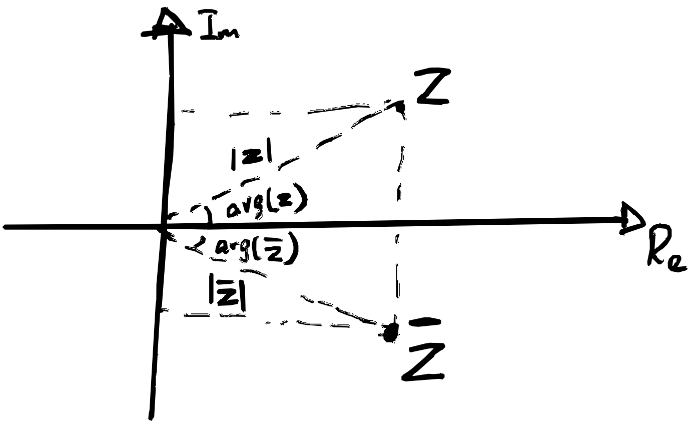


· Complex conjugate

→ $\mathrm{Im}(z) = -\mathrm{Im}(\bar{z})$
$\mathrm{Re}(z) = \mathrm{Re}(\bar{z})$ $\Rightarrow$ conjugate of z:

$z = a + jb$

$\bar{z} = a - jb$



→ $|z| = |\bar{z}|$, $\arg(z) = -\arg(\bar{z})$

Symmetrical about the real axis!

→ $z \cdot \bar{z} = (a+jb)(a-jb)$

$= a^2 - j^2 b$

$j^2 = -1$ $\quad = a^2 + b^2$

$= |z|^2$

Computational rules for complex numbers

Let $z_1 = p + jq$, $z_2 = m + jn$

Addition

$$z_1 + z_2 = (p+m) + j(q+n)$$

e.g. $(2+j) + (-5-3j)$

$= (2-5) + j(1-3)$

$= -3 - 2j$

Multiplication

$$z_1 \cdot z_2 = (p+jq)(m+jn)$$

$= pm + j^2 qn + jqm + jpn$

$= (pm - qn) + j(qm + pn)$

e.g. $(1+2j)(1+5j)$

$= (1-10) + (2+5)j$

$= -9 + 7j$

e.g. $(2+j)(2-j)$

$= 2^2 - j^2 \cdot 1$

$= 2^2 + 1^2 = 3$

Division

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|}$$

$$= \frac{(pm+qn) + j(qm-pn)}{m^2+n^2}$$

e.g. $\dfrac{1-2j}{1+j} = \dfrac{(1-2j)(1-j)}{(1+j)(1-j)}$

$= \dfrac{1-j-2j+2j^2}{2} = -0.5 - 1.5j$

Polar Form!

$$z_1 \cdot z_2 = |z_1| e^{j\arg(z_1)} |z_2| e^{j\arg(z_2)}$$

$$= |z_1||z_2| e^{j(\arg(z_1) + \arg(z_2))}$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{j(\arg(z_1) - \arg(z_2))}$$

Powers & Roots

$$Z = a + jb = |z| e^{j \arg(z)}$$

$$Z^n = |z|^n e^{j n \cdot \arg(z)}$$

$$\sqrt[n]{Z} = Z^{\frac{1}{n}} = |Z|^{\frac{1}{n}} e^{j \frac{1}{n} \arg(z)}$$

Pt. B Quadratic polynomials & factorization

$$f(x) = ax^2 + bx + c$$

quadratic equation $ax^2 + bx + c = 0$

factorization $a(x - r_1)(x - r_2) = 0$

$\Rightarrow$ quadratic formula

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

12 deriving the quadratic formula.

$$ax^2+bx+c=0 \quad \text{------------} \quad (1)$$

if we may obtain such a form:

$$(x-\beta)^2=\alpha^2 \Rightarrow x^2-2\beta x+\beta^2-\alpha^2=0 \quad (2)$$

$$\Rightarrow x-\beta=\pm\alpha$$

$$\Rightarrow x=\beta\pm\alpha \quad \text{------------} \quad (3)$$

and recall that $(x+\beta)^2=x^2+2\beta x+\beta^2$

we re-write (1) into

$$x^2+\frac{b}{a}x+\frac{c}{a}=0 \quad \text{------------} \quad (4)$$

Match (2) with (4), we get

$$-2\beta=\frac{b}{a}, \quad \beta^2-\alpha^2=\frac{c}{a}$$

$$\Rightarrow \beta=-\frac{b}{2a}$$

$$\Rightarrow \left(\frac{-b}{2a}\right)^2-\alpha^2=\frac{c}{a}$$

$$\frac{b^2}{4a^2}-\frac{c}{a}=\alpha^2$$

$$\alpha=\frac{\sqrt{b^2-4ac}}{2a}$$

in (3)

$$x=\beta\pm\alpha$$

$$=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$\square$

Quadratic equation & factorizing polynomials

$$ax^2+bx+c = a(x-r_1)(x-r_2)$$

$\forall a, b, c \in \mathbb{R}$, there exist a unique factorization in complex space $\mathbb{C}$.

- What if we have polynomials of higher orders?

→ say 3rd order, with all coefficients $\in \mathbb{R}$, we may factorize it to

$$a_3x^3+a_2x^2+a_1x+a_0 = a_3(x+r_1)\left[(x+r_2)(x+r_3)\right]$$

$$= \underbrace{a_3(x+r_1)}_{1^{st}\text{-order}} \cdot \underbrace{(x^2+(r_2+r_3)x+r_2r_3)}_{\text{quadratic}}$$

→ similarly, 4th order?

$$\sum_{i=0}^{4} a_i x^i = a_4(x^2+b_1x+b_0)(x^2+c_1x+c_0)$$

$$= a_4(x+r_1)(x+r_2)(x+r_3)(x+r_4)$$

14 The fundamental theorem of algebra

"Every non-constant polynomial in $\mathbb{C}[x]$ possesses at least 1 root in $\mathbb{C}$

THⓘ

= "$\mathbb{C}$ is an algebraically complete field"

This essentially tells us that all n-th order polynomials with complex coefficients has exactly n roots!

Proof. Consider a polynomial $P_{(x)} = \sum_{k=0}^{n} c_k x^k$

where $\forall\, k = 0, 1, \cdots, n$, $c_k \in \mathbb{C}$.

Based on THⓘ we know $P_{(x)}$ has at least 1 root, denote this root as r_1.

We find $P_{(x)} = c_n (x - r_1) \sum_{k=0}^{n-1} \frac{c_k}{c_n} x^k$

THⓘ tells us "$\sum_{k=0}^{n-1} \frac{c_k}{c_n} x^k$" also has at least 1 root.

We can apply the above procedure recursively n times then $P_{(x)}$ is completely uniquely factorized with n roots!

□

P+C. Ordinary Differential Equations (ODE)

Equation involves ordinary derivatives.

From its derivatives, find the function itself.

e.g. $\frac{dx}{dt} = \cos t \Rightarrow x = \sin t + C$

Simplist: $\frac{dx}{dt} = 0 \Rightarrow x = C$

Hmmm... a bit more complex ones:

$$\frac{dx}{dt} = \sin t + t^2$$

$$\Rightarrow \int \frac{dx}{dt} dt = \int \sin t + t^2 \, dt$$

$$\Rightarrow x = -\cos t + \frac{t^3}{3} + C$$

But.... what if we want to know C? Initial conditions!

Say $x(t_0) = K$ for $x'(t) = \sin t + t^2$

$$\Rightarrow \int_{t_0}^{t} \frac{dx}{dt} dt = \int_{t_0}^{t} \sin t + t^2 dt \Rightarrow x(t) - x(t_0)$$

$$= -\cos t + \cos t_0 + \frac{t^3}{3} - \frac{t_0^3}{3}$$

$$\Rightarrow x(t) = -\cos t + \frac{t^3}{3} + \cos t_0 - \frac{t_0^3}{3} + K$$

this is the constant.

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More complex:

$$\frac{dx(t)}{dt} = m\,x(t) + n$$

$$\Rightarrow \quad \frac{x'(t)}{m\,x(t) + n} = 1$$

$$\Rightarrow \quad \int \frac{x'(t)}{m\,x(t) + n}\,dt = \int 1\,dt$$

$$\Rightarrow \quad \int \frac{1}{m\,x(t) + n}\,dx = t + C_r$$

$$\Rightarrow \quad \frac{1}{\ }\log|m\,x(t) + n| + C_l = t + C_r$$

$$\Rightarrow \quad |m\,x(t) + n| = e^{at + aC_r + C_l}$$

$$\Rightarrow \quad x(t) = \pm\frac{e^{mC_r + C_l}}{m}\,e^{mt} - \frac{n}{m}$$

Even more ....

$$\frac{dx}{dt} + x_{(t)} = t^2$$

$$\boxed{\text{Integrating factor } e^t}$$

$$\int \left(\frac{dx}{dt} + x_{(t)}\right) dt = \int t^2 dt$$

The right part is easily approachable but the left part looks terrible.

But recall the differentiation of products:

$$\frac{d\,\gamma_{(t)}\rho_{(t)}}{dt} = \frac{d\gamma_{(t)}}{dt}\rho_{(t)} + \gamma_{(t)}\frac{d\rho_{(t)}}{dt}.$$

Hmmm... if we let $x_{(t)}$ be $\gamma_{(t)}$, if there is a function $\rho_{(t)}$ such that $\frac{d\rho_{(t)}}{dt} = \rho_{(t)}$

We would have constructed the left part to:

$$\left(\frac{dx_{(t)}}{dt} + x_{(t)}\right)\rho_{(t)}$$

Luckily we have such a $\rho_{(t)} = e^t$

Integrating by parts
$\int f_{(x)} g'_{(x)} dx = f_{(x)} g_{(x)} - \int g_{(x)} f'_{(x)} dx$

$$\Rightarrow e^t \frac{dx}{dt} + e^t x_{(t)} = e^t \cdot t^2$$

$$\Rightarrow \int \left(e^t \frac{dx}{dt} + e^t x_{(t)}\right) dt = \int e^t \cdot t^2 dt$$

$$\Rightarrow e^t x_{(t)} = e^t (2 - 2t + t^2) + C \Rightarrow x_{(t)} = 2 - 2t + t^2 + Ce^{-t}$$

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$$\int e^t t^2 dt$$
$$= t^2 e^t - \int e^t \cdot 2t\, dt$$
$$= t^2 e^t - 2te^t + \int e^t \cdot 2 dt$$
$$= (t^2 - 2t + 2) e^t + C$$

= Integrating by parts: $\int f(x) g'(x) dx = f(x) g(x) - \int g(x) f'(x) dx$

It is worth noticing:

$$\int e^{at} \cdot f(t)\, dt = \frac{1}{a} e^{at} \cdot f(t) - \int \frac{1}{a} e^{at} \cdot f'(t)\, dt.$$

Practicing:

$$\int e^{at} \sin t\, dt = \frac{1}{a} e^{at} \sin t - \frac{1}{a} \int e^{at} \cos t\, dt$$
$$= \frac{1}{a} e^{at} \sin t - \frac{1}{a} \left(\frac{1}{a} e^{at} \cos t + \int \frac{1}{a} e^{at} \sin t\, dt \right)$$

$$\Rightarrow \left(1 + \frac{1}{a^2}\right) \int e^{at} \sin t\, dt = \frac{e^{at}}{a} \left(\sin t - \frac{\cos t}{a} \right)$$

$$\Rightarrow \int e^{at} \sin t\, dt = \frac{e^{at}(a \sin t - \cos t)}{a^2 + 1}$$

If we integrate from 0 to ∞ and assume $a < 0$,

$$\Rightarrow \int_0^\infty e^{at} \sin t\, dt = \frac{1}{a^2 + 1}$$

Pt. D | Integral Transforms

Fourier & Laplace

General Integral Transform.

from m-domain to n-domain using integration.
What connects the transformation between m & n is the kernel

$$K(m,n)$$

A general transform $\mathcal{T}$

$$\mathcal{T}[f(n)] = \int_a^b f(m) K(m,n) \, dm$$

Fourier transform

kernel: $K(t,w)$

$$F(w) = \int_{-\infty}^{\infty} \frac{1}{2\pi} f(t) e^{-jwt} dt \qquad F(w) \in \mathbb{C}$$

$$f(t) = \int_{-\infty}^{\infty} F(w) e^{jwt} dw \qquad f(t) \in \mathbb{R}$$

Laplace transform

$K(t,s) = e^{-st}$

Unilateral $F(s) = \int_0^{\infty} f(t) e^{-st} dt$

Bilateral

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

$s = \sigma + jw$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} dt$$

In practice, you can get the inverse from Cauchy's Complex Residue Therom.

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We look at the unilateral Laplace transform ★ because we care about causality. ★

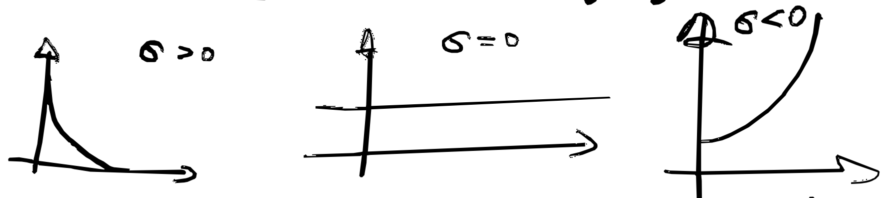
Fourier : $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$

Laplace : $\int_{0}^{\infty} f(t)\, e^{-(\sigma + j\omega)t}\, dt = \int_{0}^{\infty} f(t)\, e^{-\sigma}\, e^{-j\omega t}\, dt$

Existence of integral requires:

$f(t)\, K$ converge!

$e^{-\sigma}$



- Existence requirement for Laplace transform.
  1. $f(t)$ integrable & defined for $[0, \infty)$
  2. $f(t)$ grows slower than e^{st}

- Existence requirement for Fourier transform (Simplified)

  If and only if $f(t)$ is absolutely integrable.

$$\|x\|_1 \triangleq \int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

$e^{-\sigma}$: damping effect for convergence (the sloppy way of stating σ)

Fourier transform table.

$f(t)$	$F(\omega)$
1	$2\pi\delta(\omega)$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$e^{-at}u(t)$ $a>0$	$\frac{1}{a+j\omega}$
$te^{-at}u(t)$ $a>0$	$\left(\frac{1}{a+j\omega}\right)^2$
$e^{-a\|t\|}$ $a>0$	$\frac{2a}{a^2+\omega^2}$
$\|t\| e^{-a\|t\|}$ Re $a>0$	$\frac{4aj\omega}{a^2+\omega^2}$

2.2.1 Laplace transform table.

$f(t)$	$F(s)$
1	$1/s$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$

Properties of Laplace Transform

linearity	$af(t) \Longleftrightarrow aF(s)$ $f(t) + g(t) \Longleftrightarrow F(s) + G(s)$
Time scaling	$f(at) \Longleftrightarrow \frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$
Time shifting	$f(t-a) \Longleftrightarrow e^{-as} F(s)$
Exponential scaling	$e^{-at} f(t) \Longleftrightarrow F(s+a)$
Differentation	$f'(t) \Longleftrightarrow sF(s) - f(0)$ $f''(t) \Longleftrightarrow s^2 F(s) - s f(0) - f'(0)$ $f^{(n)}(t) \Longleftrightarrow s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - f^{(n-1)}(0)$
Integration	$g(t) = \int_0^t f(\tau) d\tau$ $\Longleftrightarrow G(s) = \frac{1}{s} F(s)$
Convolution in time is multiplication in s	$f(t) * g(t) \Longleftrightarrow F(s) G(s)$
Initial value	$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$
Final value	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$
Region of convergence	To the right of the largest real component for the poles.

Some examples:

- $f(t)=1 \quad F(s)=\int_0^{\infty} 1\cdot e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} = \frac{1}{s}$
- $f(t)=e^{at} \quad F(s)=\int_0^{\infty} e^{(a-s)t} = \frac{1}{a-s}e^{-st}\Big|_0^{\infty} = \frac{1}{s-a} \quad (s>a)$
- $f(t)=e^{(a+bi)t} \quad F(s)=\frac{1}{s-(a+bi)} = \frac{s-a+bi}{(s-a)^2+b^2}$ ------ ①

$e^a e^{bi} = (\cos b + i\sin b)e^a$, linearity property!

$$\Longrightarrow \quad \mathcal{L}[e^{at}\sin bt] = \mathrm{Im}\,① = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}[e^{at}\cos bt] = \mathrm{Re}\,① = \frac{s-a}{(s-a)^2+b^2}$$

- $n \geq 1 \in \mathbb{N}$

$f(t)=t^n \quad F(s)=\int_0^{\infty} t^n e^{-st}\,dt$ Integration by parts

$$= \frac{-1}{s}t^n e^{-st}\Big|_{t=0}^{\infty} - \int_0^{\infty} nt^{n-1}\left(-\frac{1}{s}e^{-st}\right)dt$$

$$= \frac{n}{s}\mathcal{L}[t^{n-1}]$$

we can observe that $t^n e^{-st} \to 0$ as $t \to \infty$

$t^n e^{-st} = 0$ as $t=0$

Integration by part n times gives us! $\mathcal{L}[t^0] = 1/s$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

What is really great about Laplace transform is: using $\mathcal{L}$ we can transform ODEs to polynomials!

time $\qquad$ S-domain

Integration operators: $\int dt \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \frac{1}{s}$

Differentiation operators: $\frac{d}{dt} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} s$

Example 1. Solve $\quad 0 = y' - y$

$$\xrightarrow{\text{Laplace transform}} 0 = sY - y(0) - Y$$

$$0 = (s-1)Y - y(0)$$

$$Y = \frac{y(0)}{s-1}$$

we've seen $\mathcal{L}[e^t] = \frac{1}{s-1}$

$$\Longrightarrow \quad y(t) = y(0)e^t$$

26 Example 2. Homogeneous

$$y''-2y'+y=0 \quad y(0)=1 \quad y'(0)=0$$

Let $Y(s)=\mathcal{L}[y(t)]$

$$s^2Y-sy(0)-y'(0)-2sY+2y(0)+Y=0$$

$$\Rightarrow \quad s^2Y-s-2sY+2+Y=0$$

$$\Rightarrow \quad (s^2-2s+1)Y-s+2=0$$

$$\Rightarrow \quad Y(s)=\frac{s-2}{(s-1)^2}$$

Partial fractions decomposition.

$$Y(s)=\frac{s-2}{(s-1)^2}=\frac{A}{s-1}+\frac{B}{(s-1)^2}$$

with $$=\frac{As-A+B}{(s-1)^2}$$

$$\Rightarrow \begin{cases} A=1 \\ -A+B=-2 \end{cases} \Rightarrow \quad \begin{matrix} A=1 \\ B=-1 \end{matrix}$$

$$\Rightarrow Y(s)=\frac{1}{s-1}-\frac{1}{(s-1)^2}$$

Inverse Laplace transform.

$$y(t)=e^t-te^t=(1-t)e^t$$

Example 3. Inhomogeneous.

$$y'' + y = \sin \omega t \qquad y(0) = 0 \quad y'(0) = 1 \qquad \omega \neq \pm 1$$

Laplace transform:

$$s^2 Y - s y(0) - y'(0) + Y = \frac{\omega}{s^2 + \omega^2}$$

$$\Rightarrow (s^2+1) Y - 1 = \frac{\omega}{s^2+\omega^2}$$

$$Y = \frac{\omega}{(s^2+1)(s^2+\omega^2)} + \frac{1}{s^2+1}$$

Partial fraction decomposition:

$$Y = \frac{A}{s^2+1} + \frac{B}{s^2+\omega^2} = \frac{As^2 + A\omega^2 + Bs^2 + B}{(s^2+1)(s^2+\omega^2)}$$

$$\Rightarrow \begin{aligned} A + B &= 0 \\ A\omega^2 + B &= \omega \end{aligned} \qquad \Rightarrow A = \frac{\omega}{\omega^2 - 1}$$

$$B = \frac{-\omega}{\omega^2-1}$$

$$\Rightarrow Y = \frac{1}{s^2+1} + \frac{\omega}{\omega^2-1} \cdot \frac{1}{s^2+1} + \frac{\omega}{\omega^2-1} \cdot \frac{1}{s^2+\omega^2}$$

Inverse Laplace

$$\Rightarrow y(t) = \sin t + \frac{\omega}{\omega^2-1} \sin t - \frac{1}{\omega^2-1} \sin \omega t$$

$$= \frac{\omega^2+\omega-1}{\omega^2-1} \sin t - \frac{1}{\omega^2-1} \sin \omega t$$

281 Example 4. higher order

$$y^{(4)} - 5y'' + 4y = 0 \qquad \begin{array}{ll} y(0) = 1 & y'(0) = 0 \\ y''(0) = 3 & y'''(0) = 0 \end{array}$$

Laplace transform.

$$s^4 Y - s^3 - 3s - 5s^2 Y + 5s + 4Y = 0$$

$$\Rightarrow Y = \frac{s^3 - 2s}{s^4 - 5s^2 + 4}$$

Observe denominator, 4th order polynomial but with no s^3 & s^1, the leading coefficient is 1.
We may safely factorize:

$$s^4 - 5s^2 + 4 = (s^2 + a)(s^2 + b) = s^4 + (a+b)s^2 + ab$$

$$\Rightarrow \begin{array}{l} a+b = -5 \\ ab = 4 \end{array} \Rightarrow \begin{array}{l} a = -1 \\ b = -4 \end{array}$$

$$Y = \frac{s^3 - 2s}{(s^2-1)(s^2-4)} = \frac{s^3 - 2s}{(s+1)(s-1)(s+2)(s-2)}$$

with partial fraction decomposition.

$$\Rightarrow Y(s) = \frac{\frac{1}{6}}{s+1} + \frac{\frac{1}{6}}{s-1} + \frac{1/3}{s-2} + \frac{\frac{1}{3}}{s+2}$$

$$\Rightarrow y(t) = \frac{1}{6}\left(e^{t} + e^{-t}\right) + \frac{1}{3}\left(e^{2t} + e^{-2t}\right)$$

Modelling Physical Systems

The general logic:

Physical Systems $\rightarrow$ Differential Equations $\rightarrow$ Solve $\rightarrow$ Apply (You know boundary conditions)

There can be many variations, but in this section we only take Electrical Systems & Mechanical Systems as examples. Feel free to explore more!

PtA. Electrical System

From analog circuits, we may obtain the following relationship:

	Given $i(t)$, $v(t)$?	Given $v(t)$, $i(t)$?	Given $q(t)$, $v(t)$?
Capacitor C	$\frac{1}{C}\int_0^t i(\tau)\,d\tau$	$C\frac{dv(t)}{dt}$	$\frac{1}{C}q(t)$
Inductor L	$L\frac{di(t)}{dt}$	$\frac{1}{L}\int_0^t v(\tau)\,d\tau$	$L\frac{d^2q(t)}{dt}$
Resistor R	$R\,i(t)$	$\frac{1}{R}v(t)$	$R\frac{dq(t)}{dt}$

Hmmm....

For normal RLC circuits, what really matter for characterizing these circuits are the impedances!

→ How does the circuit resist the flow of electric charges (current)?

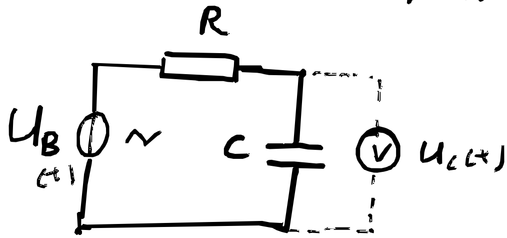
$$R = \frac{U}{I}, \quad Z = \frac{U}{I} \text{ (complex)}$$

From the above relationship, we may infer impedances in the s-domain.

		Z (impedance)
Capacitor	$\frac{1}{C}\int_0^t i(\tau)\,d\tau$	$\frac{1}{Cs}$
Inductor	$L\frac{di(t)}{dt}$	Ls
Resistor	$Ri(t)$	R

We can now analyze RLC circuits in s-domain like it's pure resistive circuit!!!

Example. RC-lowpass filter loading a capacitor.



Input: $U_B(t)$

Output $U_C(t)$

$$\Rightarrow \quad U_B(t) = U_R(t) + U_c(t)$$

$$i_B(t) = i_R(t) = i_c(t)$$

$$i_c(t) = C\frac{du_c(t)}{dt}$$

$$i_R(t) = U_R(t)/R$$

$$\Rightarrow \quad C\frac{du_c(t)}{dt} = \frac{U_B(t) - U_c(t)}{R}$$

$$\Rightarrow \quad RC\frac{du_c(t)}{dt} + U_c(t) = U_B(t)$$

Laplace

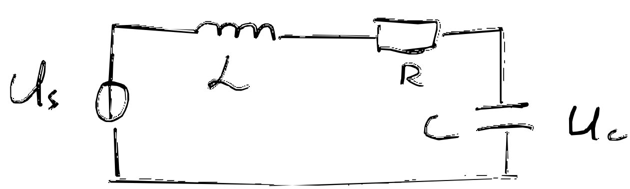
$$RCs\,U_c(s) + U_c(s) = U_B(s)$$

Transfer function: $\dfrac{U_c(s)}{U_B(s)} = \dfrac{1}{RCs+1}$

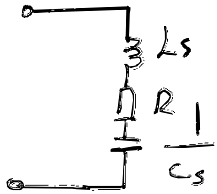
Inverse Laplace

$$\frac{U_c(t)}{U_B(t)} = e^{-\frac{t}{RC}}$$

Example.



Equiv.



Total impedance

$$Ls + R + \frac{1}{Cs}$$

The voltage over C:

$$U_c = \frac{\text{Impedance of } C}{\text{Total Impedance}} U_s = \frac{\frac{1}{Cs}}{Ls+R+\frac{1}{Cs}} U_s$$

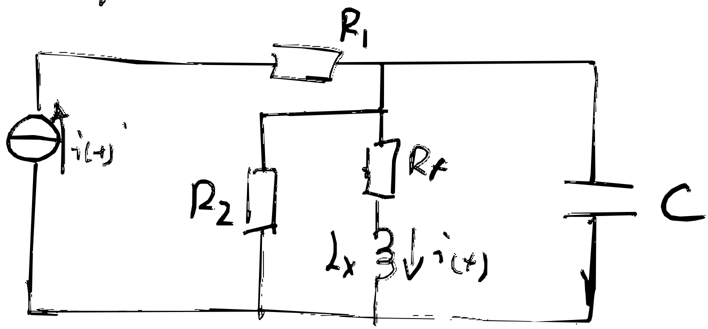
$$\Rightarrow \frac{U_c}{U_s} = \frac{\frac{1}{Cs}}{Ls+R+\frac{1}{Cs}} = \frac{1}{LCs^2+RCs+1}$$

Transfer function.

More complex example.

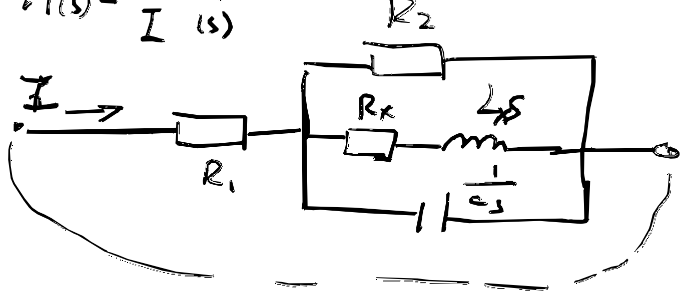
Input: $i(t)$

Output: $i_x(t)$



Find transfer function $H(s) = \frac{I_x(s)}{I(s)}$

Equiv. Circuit.



Relationship:

$$I = I_{R_2} + I_x + I_c$$

$$U_{R_2} = U_c = U_{R_x} + U_{L_x}$$

$$\Rightarrow \quad U_c = I_{R_2} R_2 = (R_x + L_x s) I_x = \frac{I_c}{sC}$$

$$\Rightarrow \quad I_{R_2} = \frac{R_x + sL_x}{R_2} I_x \,, \quad I_c = sC(R_x + sL_x) I_x$$

$$\Rightarrow \quad I = \left(\frac{R_x + sL_x}{R_2} + 1 + sC(R_x + sL_x) \right) I_x$$

$$\frac{I_x}{I} = \frac{R_2}{R_2 L_x C s^2 + (L_x + R_x R_2 C) s + R_x + R_2}$$

Pt B. Mechanical Systems

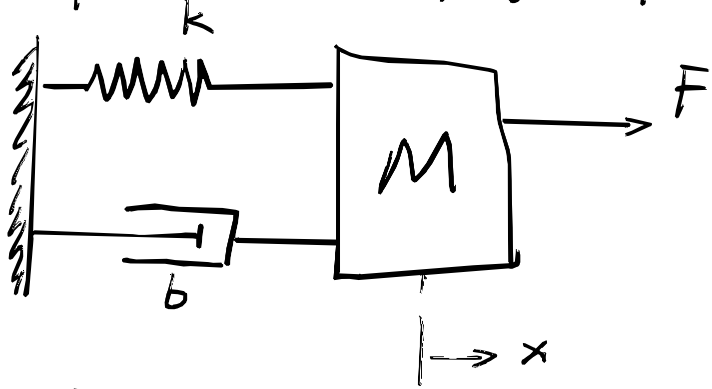
Similar to what we have seen in electrical systems for mechanical systems, we have:

	Force-Velocity	Force-displacement
Damper b $F=bv$	$F=bv$	$F=b\frac{dx(t)}{dt}$
Spring k $F=kx$	$F=k\int_0^t v(\tau)\,d\tau$	$F=kx$
Mass (Inertia) m $F=ma$	$F=m\frac{dv(t)}{dt}$	$F=m\frac{d^2x(t)}{dt^2}$

"Impedance form"

	F-v Impedance	F-x impedance
Damper	b	bs
Spring	$\frac{k}{s}$	k
Mass	ms	ms^2

Example: classical spring-damper-mass system.

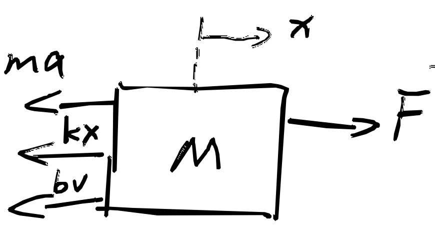


Velocity when force applied?

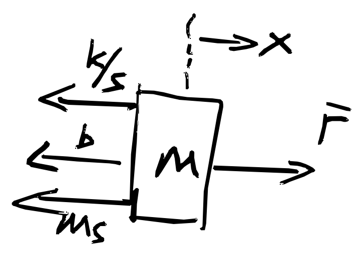
Transfer function $\dfrac{V}{F}$

Free body diagram

Force-Velocity impedance



$\Rightarrow$



$$F = \left(\frac{k}{s} + b + Ms\right) V$$

$$\Rightarrow \frac{V}{F} = \frac{s}{Ms^2 + bs + k}$$

361 Transfer Functions : P & Z

$$\begin{tikzcd} \text{Physical System} \arrow[r, dashed] \arrow[d] & \text{Experimental DATA} \arrow[dd, dashed, "\text{Approximation}"] \\ \text{ODE} \arrow[dr, "\mathcal{L}", "\text{Block diagram}"'] & \\ & \boxed{\text{Transfer Function}} \end{tikzcd}$$

To further understand the physical system, we need to really understand what's happening in the transfer function.

Example

$$y'' + 2y' - 8 = 3x' + 1x$$, assume 0 initial conditions

$$\xrightarrow{\mathcal{L}} s^2Y + 2sY - 8Y = 3sX + X$$

(Open loop) $\Longrightarrow$ TF: $$H(s) = \frac{Y(s)}{X(s)} = 3\,\frac{s+\frac{1}{3}}{s^2+2s-8}$$

$$\xlongequal{\text{factorize}} 3\,\frac{s+1/3}{(s+4)(s-2)}$$

"Physical systems" described by "linear inhomogeneous ordinary differential equations"

through Laplace transform, we must obtain

transfer functions $H(s) = K \frac{\text{polynomial } N(s)}{\text{polynomial } D(s)}$ with <u>real</u> coefficients

"the fundamental therorem of algebra" (P17)

Guarantees us that we may have a unique decomposition of $H(s)$

$$H(s) = K \frac{\prod_{i=1}^{n} (s - z_i)}{\prod_{j=1}^{m} (s - p_j)} = \frac{N(s)}{D(s)}$$

DEFINE: $\boxed{\text{Poles}}$: roots of $D(s)$

$\boxed{\text{Zeros}}$: roots of $N(s)$

In the example above, we have n zeros $\{z\}$, m poles $\{p\}$.

Due to we have real coefficients, thus.

- either all poles & zeros are real
- or complex poles & zeros alway appear in conjugate pairs

Back to our perious example

$$H(s) = 3\frac{s+1/3}{(s+4)(s-2)}$$

we can find poles zeros according to our standard factorized form of $H(s) = k\dfrac{\prod(s-z_i)}{\prod(s-p_j)}$

➡ gain $K = 3$, poles: $-4, 2$, zero: $-1/3$

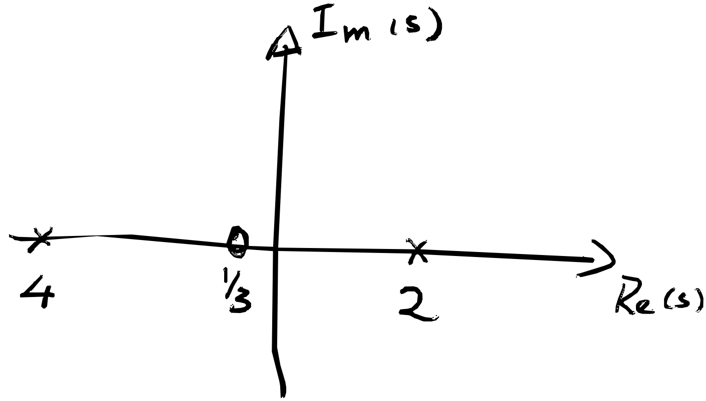
$\boxed{\text{Properties!}}$

$s =$ pole	$H(s) \to \infty$
$s =$ zero	$H(s) \to 0$

Visualizing poles & zeros in "a complex plane"

a complex plane in s-domain

notation:
zeros: "o"
poles: "x"



Continuing with our example:

$$H(s) = 3\,\frac{s+\frac{1}{3}}{(s+4)(s-2)}$$

$$\xRightarrow{\text{partial fraction decomposition}}$$

$$H(s) = \frac{11/6}{s+4} + \frac{7/6}{s-2}$$

$$\xrightarrow{\mathcal{L}^{-1}} h(t) = \frac{1}{6}\left(11e^{-4t} + 7e^{2t}\right)$$

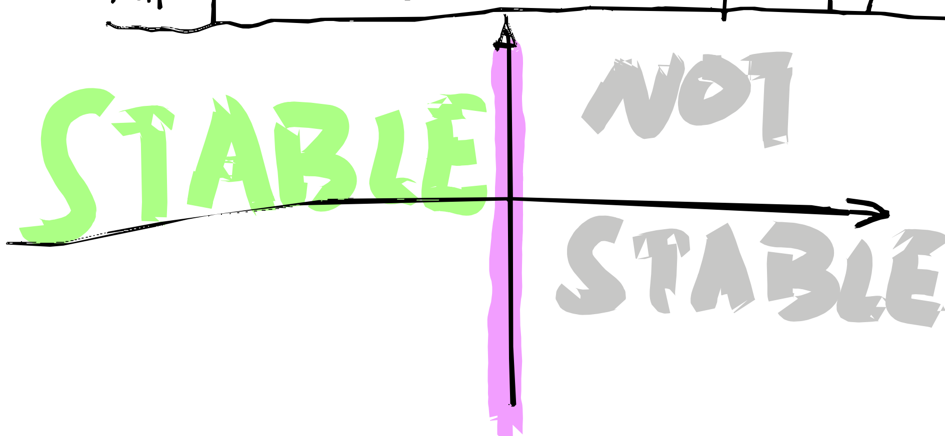
poles: -4 $\quad$ 2

Stability requirement $\displaystyle\int_0^{\infty} |h(t)|\,dt < \infty$!

as $t \to \infty$, $h(t) \to 0 + \infty = \infty$

Thus, naturally, we derive our stability requirement:

<u>All poles need to be in the *left half plane* !</u>



4d Hmmm......

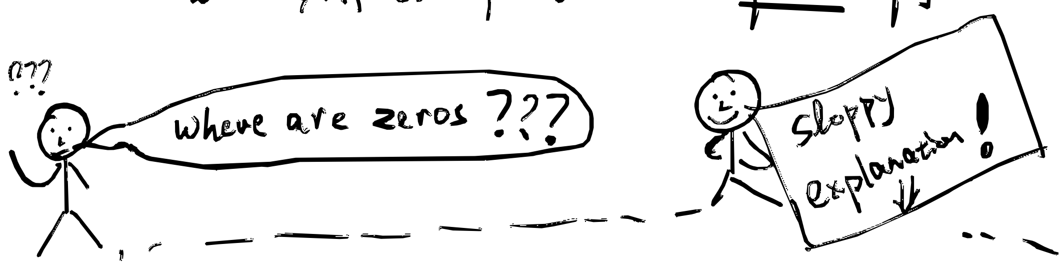
Laplace transform solving ODE .....

The physical system $h(t)$ is decomposed into:

the addition of linearly independent exponentials

$$h(t) = \sum_{\lambda} C e^{\lambda t}$$

where $\{\lambda\}$ corresponds to the poles $\{p\}$



Zeros are where these exponentials happens to be cancelling each other in the frequency domain.

- when $s =$ zero, $|H(s)| = 0$
- Through the partial fraction decomposition you see, where you change "A, B, C, D..." On the numerators to match the originals. When you change "C" in $h(t) = \sum_{\lambda} C e^{\lambda t}$, the numerators in the partial fraction decompositions change as well, such that they cancell each other else where or just don't cancell!

Furthing looking at $h(t)$:

$$h(t) = \sum C e^{\lambda t} = \sum C e^{\sigma t} e^{j\omega t}$$

! Recall $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

For poles $\{p_i = \sigma_i + j\omega_i\}$:

$Re(p_i) = \sigma_i$ determines the decay (damping) !

$Im(p_i) = \omega_i$ determines the oscillation !

We have just visualized poles & zeros in "a complex plane" by marking poles as "x" and zeros as "o"

① Which complex plane?

Why?

② Are these just conventions?

③ We have a transfer function $H(s)$ with $s = \sigma + j\omega$, we only plotted poles & zeros in $\mathbb{C}$.

That is just "s", where is "$H(s)$" ???

The variable

Next page, we answer these 3 questions.

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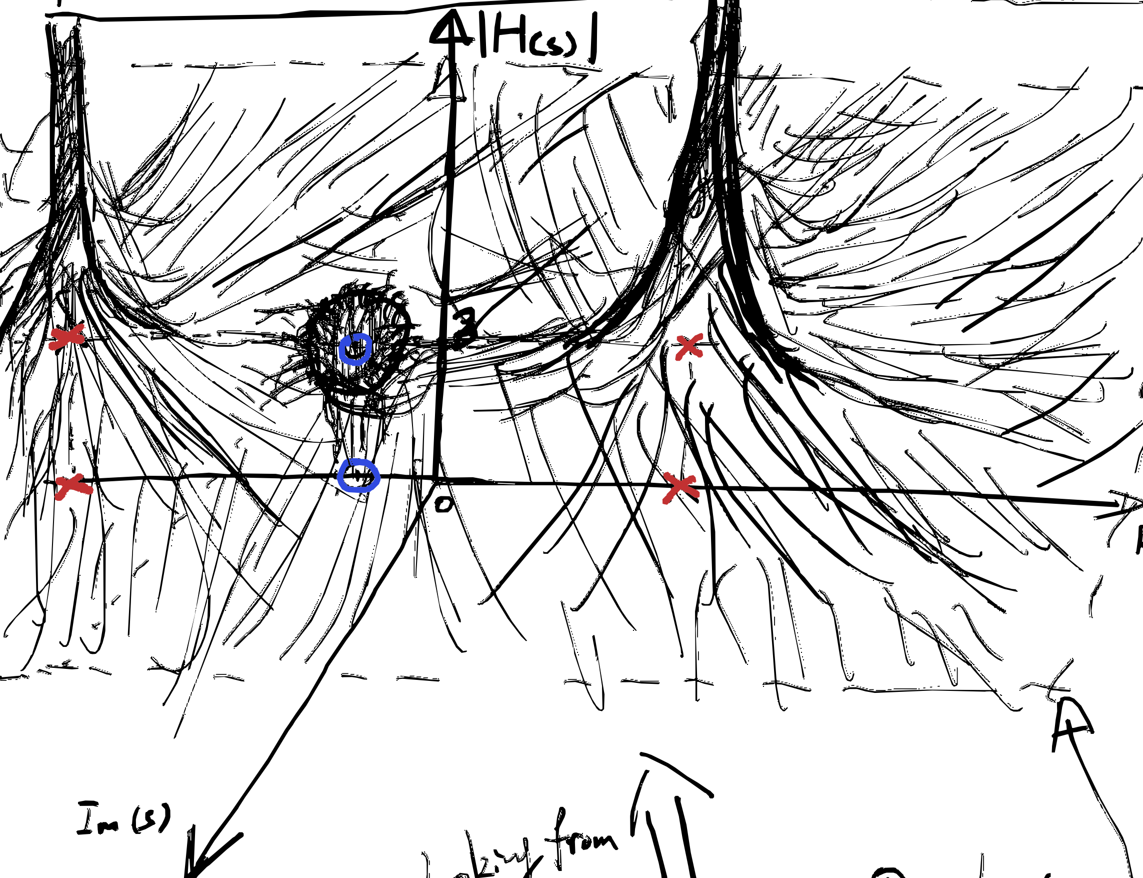
$$H(s) = 3\,\frac{(\frac{1}{3}s+1)}{(\frac{1}{4}s+1)(\frac{1}{2}s-1)}$$

We have $s = \sigma + j\omega$, if we plot $|H(s)|$, we need a 3D-plot!

③

Poles in 3D visualization do look like "poles"

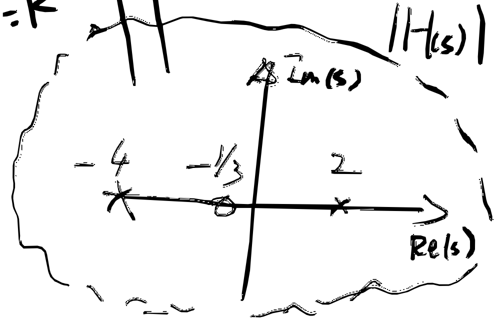


② Looking "up" from bottom, we have "poles" going "into" the plane $|H(s)| = K$, so ⊗

we have "zeros" going "out of" the plane $|H(s)| = K$, so ⊙

Looking from underneath $|H(s)| = K$

① Level $|H(s)| = K$
$= 3$



In addition, we may look at an interesting case:

the laplace transfrom of a periodic function

Say: a general periodic function $y(t)$ in one period T.

The periodic function $f(t)$ contains infinite delayed copies of $y(t)$.

$$y(t) = \begin{cases} f(t) & 0 < t < T \\ 0 & \text{otherwise}. \end{cases}$$

$$f(t) = y(t-0T) + y(t-T) + y(t-2T) + \cdots$$

$$= \sum_{k=0}^{\infty} y(t-kT)$$

We apply Laplace Transfrom:

$$F(s) = Y(s) \sum_{n=0}^{\infty} e^{-nTs}$$

$\sum_{n=0}^{\infty} e^{-nTs}$ is a geometric series, thus:

$$F(s) = \frac{1}{1-e^{-sT}} Y(s)$$

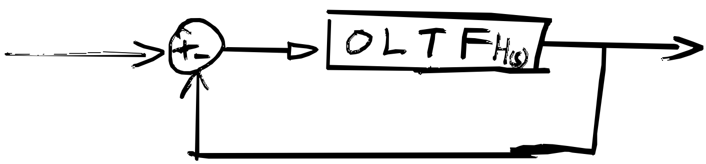
$\boxed{\frac{1}{\text{period}}} \Longrightarrow$ Frequency!

we have created a pole at the sampling frequency $\frac{1}{T}$.

4.4 Root Locus technique

what happens in closed loop system

We were looking at the transfer function of the "process" and understanding physical systems. Recall · we had a controll loop with unity feedback.



$$H(s) = K\frac{N(s)}{D(s)}$$

OL poles: $D(s)$ roots

OL zeros: $N(s)$ roots

Closed loop transfer function

$$\frac{H(s)}{1+H(s)} = \frac{K\frac{N}{D}}{1+K\frac{N}{D}} = \frac{KN(s)}{KN(s)+D(s)}$$

CL poles: $KN(s)+D(s)$ roots

CL zeros: $KN(s)$ roots → The same with OL

As we close the control loop, K increase from 0 to ∞:

Zeros stay!

Poles move!

↳ Dependant on K value.

How might poles move as we close the control loop & increase the gain of the controller? ☆

Intuitively, $CLTF = \dfrac{KN(s)}{D(s) + KN(s)}$

As $K = 0$, $D(s)$ dominates, poles at OLTF poles!

$K \uparrow$, $N(s)$ start to affect the poles

As $K \to \infty$, $N(s)$ dominates, poles at OLTF zeros!

- Should CLTF poles move from OLTF poles to OLTF zeros as K increase from 0 to ∞ ???

→ We look at a general example of a 2nd-order system.

$$H(s) = \frac{K}{as^2 + bs + c} \quad \text{(OLTF)} \quad \left(\text{with no zero}\right)$$

characteristic equation: $as^2 + bs + c = 0$

OL poles: $p_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\text{(CLTF)} \quad \frac{H(s)}{1 + H(s)} = \frac{K}{as^2 + bs + (c + K)}$$

characteristic equation: $as^2 + bs + (c+K) = 0$

$$\Rightarrow \text{CL poles } p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a(c+K)}}{2a}$$

4.6.1 CLTF poles

$$p_{1,2} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4a(c+K)}}{2a}$$

Let's just assume: OLTF poles are 2 distinct real values

$$\Rightarrow b^2 > 4ac$$

- when $K = 0$

$$\text{CLTF poles} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \text{OLTF poles}$$

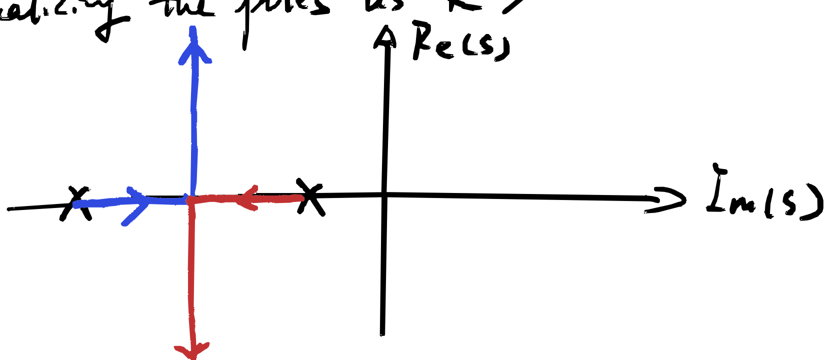
- As $K \uparrow$, till a point where $b^2 = 4a(c+k)$
  CLTF poles converge to 1 single real value

- As $K \uparrow$, we have complex poles!

$$Re(p_{1,2}) = -\frac{b}{2a}$$

$$Im(p_{1,2}) = \pm \frac{\sqrt{b^2 - 4a(c+k)}}{2a}$$

Visualizing the poles as $k \uparrow$



What if we have a zero?

$\boxed{\text{zero: } -z}$

OLTF $\quad H(s) = k\dfrac{s+z}{as^2+bs+c}$

CLTF $\quad \dfrac{H(s)}{1+H(s)} = \dfrac{k(s+z)}{as^2+(b+k)s+(c+kz)}$

characteristic equation CLTF:

$$as^2+(b+k)s+(c+kz)=0$$

$$\Rightarrow p_{1,2} = \frac{-(b+k)\pm\sqrt{(b+k)^2-4a(c+kz)}}{2a}$$

- $k=0$, $\quad CL\ p_{1,2} = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a} = OL\ p_{1,2}$

- $k\to\infty$

$\to\infty$ $\quad \to -\infty$ (square, increase a lot faster)

$$\lim_{k\to\infty}(CL\ p_1) = \lim_{k\to\infty}\frac{-(b+k)-\sqrt{(b+k)^2-4a(c+kz)}}{2a}$$

$$= -\infty$$

$$\lim_{k\to\infty}(CL\ p_2) = \lim_{k\to\infty}\frac{-(b+k)+\sqrt{(b+k)^2-4a(c+kz)}}{2a}$$

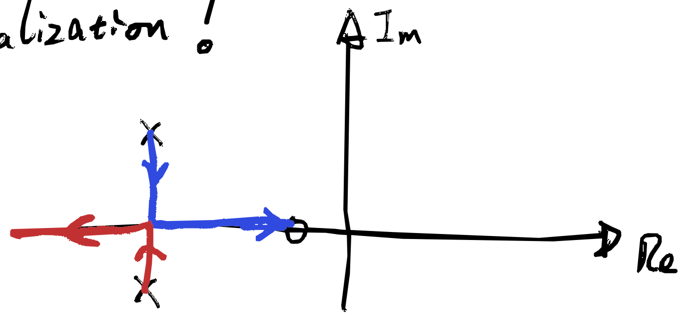
$$= \lim_{k\to\infty}\frac{(b+k)^2-(b+k)^2+4a(c+kz)}{2a\left[-(b+k)-\sqrt{(b+k)^2-4a(c+kz)}\right]}$$

$$= \lim_{k\to\infty}\frac{4ac+4akz}{2a\left[-(b+k)-\sqrt{(b+k)^2-4a(c+kz)}\right]} = \lim_{k\to\infty}\frac{\frac{4ac}{k}+4az}{\left[-\left(\frac{b}{k}+1\right)-\sqrt{\frac{b^2}{k^2}+\frac{2b}{k}+1-\frac{4ac}{k^2}-\frac{4az}{k}}\right]2a}$$

$$= \frac{4az}{(-2)2a} = -z$$

(OUR ZERO)

Visualization !



In general, we may discover that,

as K increase from 0 to ∞,

CLTF poles move from OLTF poles to finite zeros or infinity.

Hmmm.... we know the start point & end point of the CLTF poles. But how does the poles move exactly between the start & end ?

Asymptotes !

(We find that using polar view)

Asymptotes of CLTF pole movements.

OLTF $KH(s)$ CLTF $\dfrac{H(s)}{1+KH(s)}$

Recall polar representation of $H(s) = |H(s)| \angle (H(s))$

Characteristic equation CLTF: $1+KH(s) = 0$

$$\Rightarrow H(s) = -\frac{1}{K}$$

$$\Rightarrow |H(s)| = \frac{1}{K}, \quad \angle(H(s)) = 180^\circ$$

As $K \to \infty$, $|H(s)| \to 0$, $\angle(H(s)) \longrightarrow 180^\circ$

To not break this property, the asymptotes should be symmetric about the real axis.
(Also recall complex conjugte root pair property)
And if multiple extra poles exist, the routes of the CLTF poles should "uniformly scan the entire $\mathbb{C}$".

\# asymptotes of CL poles to infinity

$$= \#P - \#Z$$

Angles separation between two CL pole "routes" next to each other $\dfrac{360^\circ}{\#P - \#Z}$

Centroids of asymptotes:

$$\beta = \frac{\sum p - \sum z}{\# p - \# z}$$

Then what is Root Locus?

➤ Root Locus describes how CLTF poles move when then controller gain K increase from zero to infinitly large.

We have done somuch analysis so that we can just summarize to properties of root locus.

- Start at OLTF poles, end at OLTF zeros.
- Number of branches (routes) = number of OLTF poles
- Root Locus on real axis:

to the left of an odd number of distinct real axis poles & zeros

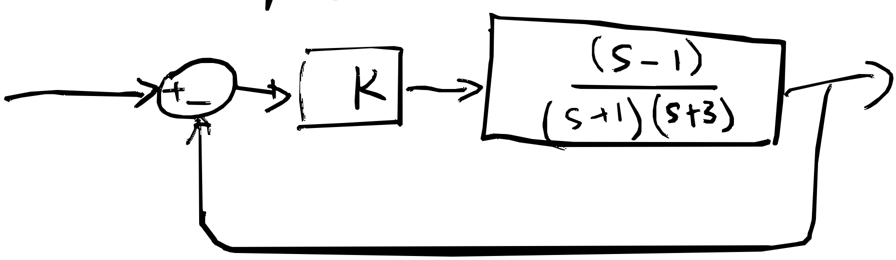
- Shape: asymptotes angles $\phi_l = \frac{(2l+1)180^\circ}{\# p - \# n}$

Centroids $\beta = \frac{\sum p - \sum n}{\# p - \# n}$

If $x=r$ is a double root of polynomial $P(x)$, then $P(r)=0$ and $\left.\frac{dP(x)}{dx}\right|_{x=r}=0$.

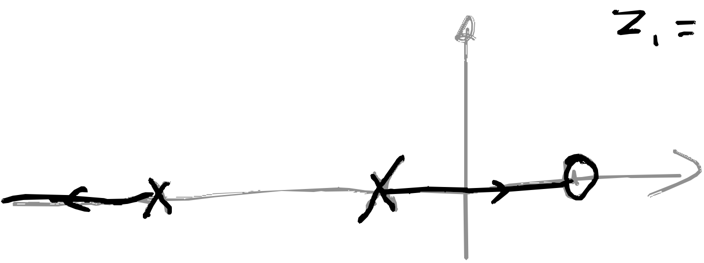
- Break in/out points: solve $\frac{d(1+kL(s))}{ds}=0$

Using root locus technique, we may design the closed loop gain controller, for example:



Poles & zeros: $p_1=-1$, $p_2=-3$

$z_1=1$



We can see that as K grows larger, there will be a pole going into the RHP, causing the closed-loop system to go unstable

...... But when?

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$$\frac{(s-1)}{(s+1)(s+3)}$$

$$\text{CLTF} = \frac{(s-1)}{(s+1)(s+3)+k(s-1)}$$

characteristics polynomial:

$$s^2+4s+3+ks-k=0$$

$$s^2+(4+k)s+3-k=0$$

Roots:
$$\frac{-4-k\pm\sqrt{(4+k)^2-4(3-k)}}{2}$$

$$=-\frac{4+k}{2}\pm\frac{\sqrt{(4+k)^2-4(3-k)}}{2}$$

Boundary condition: one root is zero

K 0 → ∞, hence we look at:

$$-(4+k)+\sqrt{(4+k)^2-4(3-k)}=0$$

$$(4+k)^2-4(3-k)=(4+k)^2$$

$$-4(3-k)=0$$

$$k=3$$

when $k=3$, the system has a pole at the origin and it's marginally stable.

when $k>3$, the system becomes unstable.

A peak into transfer functions

Transfer function $H(s)$

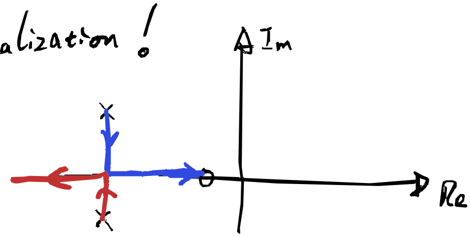
Domain $H(s)$: $\mathbb{C}$

Co-domain $H(s)$: $\mathbb{C}$

So, if we would like to visualize $H(s)$, we need 4-dimensional graph!

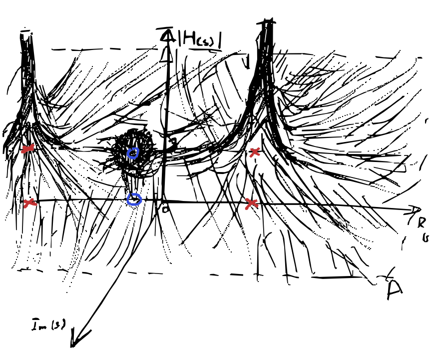
We live in 3d space, not nice if we need to visualize in 4d. So, we have developed the root-locus technique to observe the critical points in the domain of $H(s)$.

Visualization!



← s-plane.

$s = \sigma + j\omega$



← 3d s-plane with $|H(s)|$

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what's missing?

If we write $H(s)$ in polar form:

$$H(s) = |H(s)| \angle (H(s))$$

we missed the phase part

what's related to phase ???

Frequency!

In life we do observe the effect of different frequency, when you are riding a horse, you need to adjust your body according to the speed of the horse.

If the horse is riding slow, with small steps. the vertical vibration frequency is high.

If the horse is riding fast, with large steps. the vertical vibration frequency is low.

I didn't know how to move my body to cope with the horse when I was little, so I find it easier when the horse is running compared to walking.

Frequency Domain Analysis.

We discussed controller gain K, but frequency also plays a large part in control systems.

Frequency response occurs when a process receives a periodic signal.

Assume the input is $x_{(t)} = A\sin(\omega t)$ for $t \geq 0$ (causal), the frequency response of H

$$y_{(t)} = x_{(t)} * h_{(t)}$$

For convenience, we use Laplace transform:

$$Y_{(s)} = X_{(s)} H_{(s)} = \frac{A}{s^2+\omega^2} H_{(s)}$$

Think about the fundamental theorem of algebra and the partial fraction decomposition we have applied in previous sections, we do partial fraction decomposition to $Y_{(s)}$:

56| $Y(s) = X(s)H(s) = \frac{A}{s^2+w^2}$

$$= \underbrace{\frac{M}{s+jw} + \frac{N}{s-jw}}_{} + \underbrace{\left\{ \text{partial fraction decomposition of } H(s) \right\}}_{}$$

Depends largely on the input, called "forced response"

Depends largely on the system $H(s)$, called "natural response".

The next step of partial fraction decomposition is to find "M" "N" and all the coefficients on the numerators.

Based on the uniqueness of Laurent series, M is the coefficient of $\frac{1}{s+jw}$ in the Laurent series expansion of $Y(s)$ about the singularity point $s=-jw$.

Using Cauchy's residue theorem,

$$M = \mathrm{Res}\left(Y(s), -jw\right)$$

and $s=-jw$ is a simple root, we write

$$H(s) = \frac{Z(s)}{P(s)}$$

where $Z(s)$ and $P(s)$ represent the polynomials on the numerator and denominator.

$$M = \mathrm{Res}\left(Y(s), -jw\right) = \left.\frac{A Z(s)}{\frac{d\,(s^2+w^2)P(s)}{ds}}\right|_{s=-jw}$$

$$= \frac{A Z(-jw)}{-2jw\,P(-jw)} = \frac{jA}{2w} H(-jw)$$

Since $H(s)$ is symmetrical about the real axis, thus $H(-jw) = H(jw)$.

$$M = \frac{jA}{2w} H(jw)$$

Similarily, we may find

$$N = -\frac{jA}{2w} H(jw) = \bar{M}$$

the complex conjugate of M

Then, for the forced response:

$$Y_{forced} = \frac{M}{s+j\omega} + \frac{\bar{M}}{s-j\omega}, \quad M = \frac{jA}{2\omega} H(j\omega)$$

$$= \frac{(s-j\omega-(s+j\omega))\frac{jA}{2\omega}H(j\omega)}{s^2+\omega^2}$$

$$= \frac{A}{s^2+\omega^2} H(j\omega)$$

We have recovered $X(s)$ from the forced response!

The frequency response of the system can thus be found via $H(j\omega)$!

This property also implies that if there is a system $H(s)$ that we don't know the transfer function, we may probe the transfer function by scanning through the entire frequency range!

$\omega: 0 \longrightarrow \infty$

But how?

We have $H(s) = \frac{Z(s)}{P(s)}$ where $Z(s)$ and $P(s)$ are both polynomials with real coefficients.

Thanks to the fundamental theorem of algebra, we may write:

$$Z(s) = k_1 (s+z_1)(s+z_2) \cdots$$

$-z_1, -z_2, -z_3 \ldots$ are roots of $Z(s)$

$$P(s) = k_2 (s+p_1)(s+p_2)(s+p_3) \cdots$$

$-p_1, -p_2, -p_3$ are roots of $P(s)$

Still, everything is multiplied together, which means it's difficult to examine the effects of each decomposed parts.

But, we have an elementary function that changes multiplication to addition:

the logarithmic function.

$\boxed{\text{Decibels:}}$

$$\text{dB} = 20 \log_{10} \text{linear}, \quad \text{linear} = 10^{\frac{\text{dB}}{20}}$$

Multiplication in linear scale

$\Updownarrow$

Addition in logarithmic scale

We further organize $H(s)$ a bit:

$$H(s) = \frac{k_1 (s+z_1)(s+z_2)\cdots}{k_2 (s+p_1)(s+p_2)\cdots}$$

$$= \frac{k_1 \prod_m z_m}{k_2 \prod_n p_n} \frac{\left(\frac{s}{z_1}+1\right)\left(s/z_2+1\right)\cdots}{\left(\frac{s}{p_1}+1\right)\left(s/p_2+1\right)\cdots}$$

Why do we do this?

Think about circuit analysis, if we have a periodic AC electric signal as input, when we look at the frequency response of $H(s)$ by letting $s=j\omega$, we have $\omega>0$. If we have a DC signal input, then the frequency $\omega=0$. Only the gain below remains:

$$H(j0) = \frac{k_1 \prod_m z_m}{k_2 \prod_n p_n} = K_{DC}$$

We call this gain the $\boxed{\text{DC gain}}$

Now

$$H(s) = K_{DC}\frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\cdots}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\cdots}$$

We look at the logarithmic scale of $H(s=j\omega)$

Magnitude:

$$|H(j\omega)| = 20\log_{10}|K_{DC}| + \sum_m 20\log_{10}\left|\frac{j\omega}{z_m}+1\right|$$
$$- \sum_n 20\log_{10}\left|\frac{j\omega}{p_n}+1\right|$$

Phase:

$$\angle H(j\omega) = 0 + \sum_m \angle\left(\frac{j\omega}{z_m}+1\right)$$
$$- \sum_n \angle\left(\frac{j\omega}{p_n}+1\right)$$

We see that we can separately assess the effect of poles & zeros and add the together when we scan the frequency from $0 \to \infty$.

6.2

- $\left(\frac{j\omega}{z}+1\right)$ - - - - - a zero $-z$

$$Re\left(\frac{j\omega}{z}+1\right) = 1 \qquad \omega : 0 \to \infty$$

$$Im\left(\frac{j\omega}{z}+1\right) = \frac{\omega}{z}$$

$$\left|\frac{j\omega}{z}+1\right| = \sqrt{\frac{\omega^2}{z^2}+1} \leftarrow \text{(Magnitude)}$$

when $\omega \ll z$, $\left|\frac{j\omega}{z}+1\right| \to 1$, $\angle\left(\frac{j\omega}{z}+1\right) \to 0^\circ$

when $\omega = z$, $\left|\frac{j\omega}{z}+1\right| \to \sqrt{2}$, $\angle\left(\frac{j\omega}{z}+1\right) \to 45^\circ$

when $\omega \gg z$, $\left|\frac{j\omega}{z}+1\right| \to \infty$, $\angle\left(\frac{j\omega}{z}+1\right) \to 90^\circ$

- $\frac{1}{\left(\frac{j\omega}{p}+1\right)}$ - - - - - - a pole $-p$

$$\frac{p}{j\omega+p} = \frac{p(j\omega-p)}{(j\omega+p)(j\omega-p)} = \frac{p(p-j\omega)}{\omega^2+p^2}$$

$$Re\left(\frac{1}{\left(\frac{j\omega}{p}+1\right)}\right) = \frac{p^2}{\omega^2+p^2}$$

$$Im\left(\frac{1}{\left(\frac{j\omega}{p}+1\right)}\right) = \frac{-\omega p}{\omega^2+p^2}$$

$$\left|\frac{1}{\left(\frac{j\omega}{p}+1\right)}\right| = \frac{p}{\omega^2+p^2}\sqrt{\omega^2+p^2} \leftarrow \text{(Magnitude)}$$

when $\omega \ll p$, magnitude $\to 1$, phase $\to 0^\circ$

when $\omega = p$, magnitude $\to \sqrt{2}/2$, phase $\to -45^\circ$

when $\omega \gg p$, magnitude $\to 0$, phase $\to -90^\circ$

Effects summary

	Phase	Magnitude
Pole p	Clockwise $90°$	Suppress $\omega > p$
Zero z	Counter clockwise $90°$	Boost $\omega < z$
DC gain k_{DC}	No effect	Offset at $20\log_{10}(k_{DC})$

The magnitude turning points at $\omega = p$ and $\omega = z$ are called <u>corner frequencies</u>.

Suppress and boost, how fast?

$\omega >> z$ $\quad \left|\frac{j\omega}{z} + 1\right| = \sqrt{\frac{\omega^2}{z^2} + 1} \approx \frac{\omega}{z}$

log scale

$$20\log_{10}\left(\frac{\omega}{z}\right) = 20\log_{10}(\omega) - 20\log_{10}(z)$$

take the derivative, we find rate of change: 20 dB (boosting)

Similarily, for $\omega >> p$, rate of change -20 dB (suppressing)

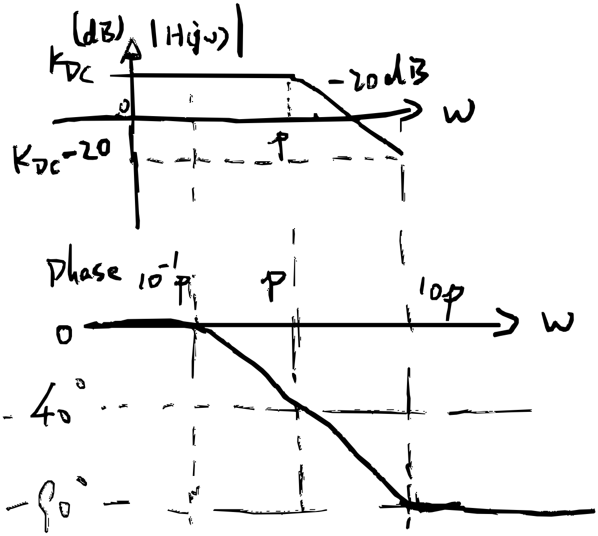
64 Can we draw these effects?

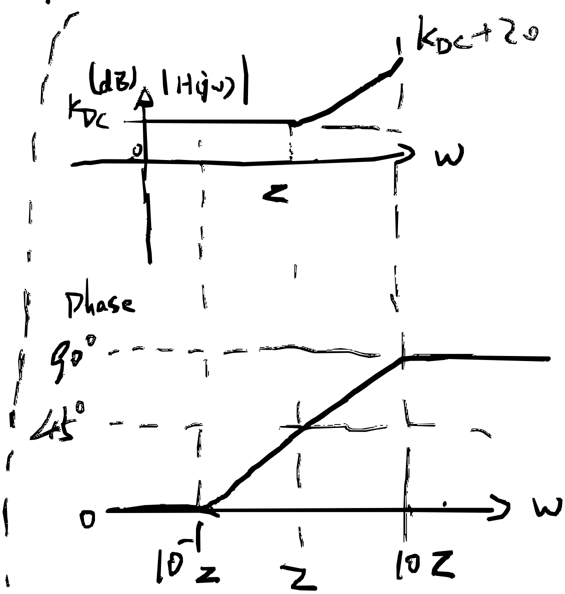
Yes! Bode plots!

2 plots: magnitude plot, phase plot

Additive effect! Log-log scale

Asymptotes pole at "p", zero at "z"





Example

$$H(s) = 1000 \frac{s(s+10)}{(s+100)^2(s+1)}$$

$$K_{DC} = 0.1 = 20\log_{10}(0.1) = -20\,dB$$

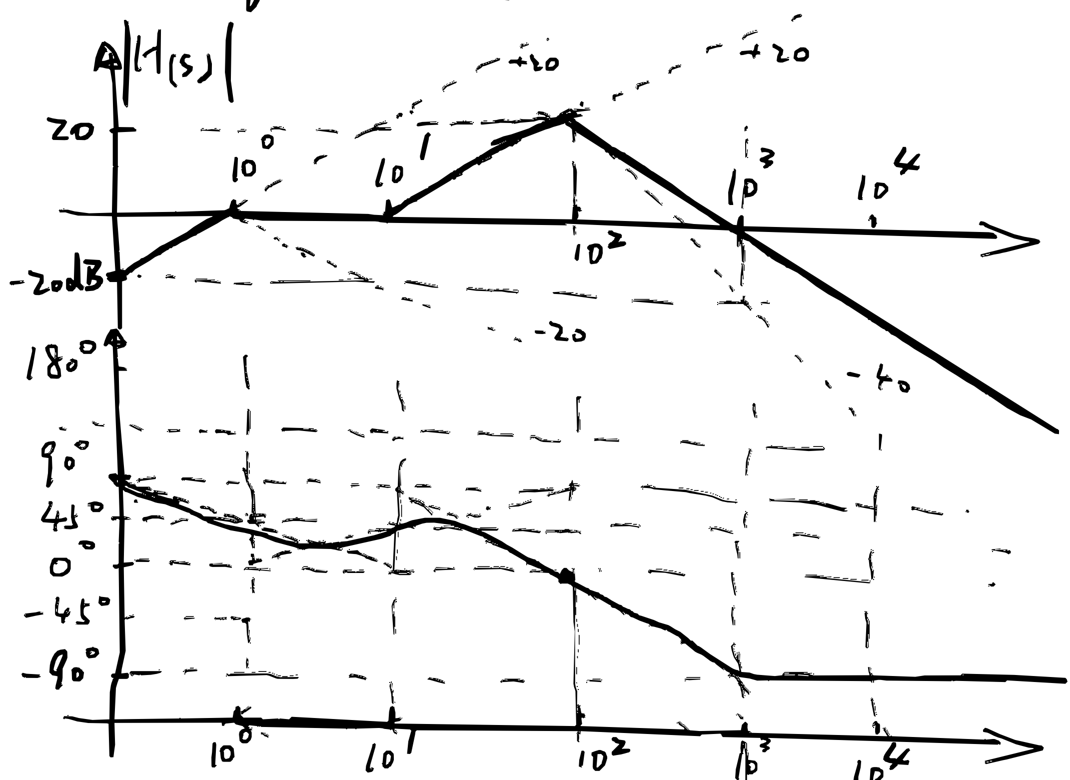
Corner frequencies

ω	0	1	10	100
slope	+20	-20	+20	-40
phase	+90°	-90°	+90°	-180

+20 0 +20 -20

Starting phase +90°, starting slope +20dB

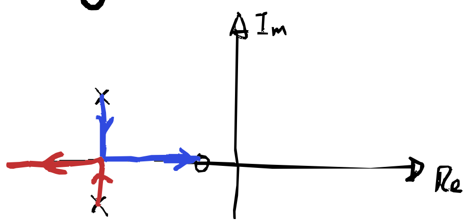
Starting magnitude $K_{DC} = -20dB$



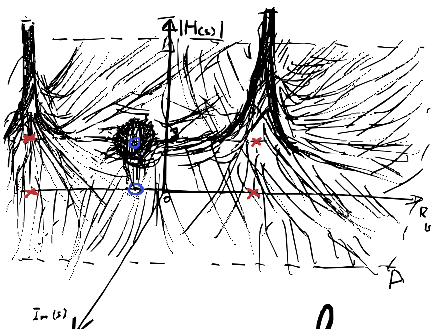
Till now we can pause a bit and think about the visualization of the transfer function $H(s)$.

On page 53, we discussed the domain and co-domain of $H(s)$.

With pole-zero plot & root locus, we can investigate the system behaviour in s-plane.



← s-plane.
$s = \sigma + j\omega$



← 3d s-plane with $|H(s)|$

Observe the frequency response $H(j\omega)$, it appears that we have yet another opportunity to visualize $H(j\omega)$ in a 3D plane.

Further more, like root-locus, we didn't have to plot the 3rd axis, we just need to observe the trajectory in a 2D plane. We select the co-domain $\mathbb{C}$ in which we visualize $Im(H(j\omega))$ and $Re(H(j\omega))$.

This would be the

Nyquist Plot

A Nyquist plot shows on the complex plane the real part of a frequency response function against its imaginary part. The frequency ω is an implicit variable.

Example $\quad KH(s) = \dfrac{K}{s+1}$

$$KH(j\omega) = \frac{K}{j\omega + 1} = \frac{K(1-j\omega)}{\omega^2+1}$$

$$= \frac{K}{\omega^2+1} - j\frac{K\omega}{\omega^2+1}$$

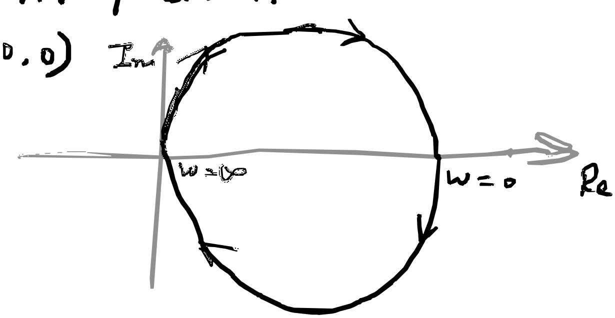
$\omega = 0 \qquad K - 0j$

$\omega \uparrow \qquad Im(KH(j\omega)) \uparrow$

the trajectory rotates clockwise from the real axis to the 4th quadrant.

$\omega \to \infty \qquad KH(j\omega) \to (0,0)$

We can obtain the trajectory $\longrightarrow$ on the right.



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We make use of Cauchy's argument principle.

First, for our convenience, we define

- A function $f: G \to \mathbb{C}$ is a <u>meromorphic function</u> in G if G is open and f is defined and analytic in G except for the poles.

- A function $f: G \to \mathbb{C}$ is <u>analytic</u> if f is continuously differentiable on G.

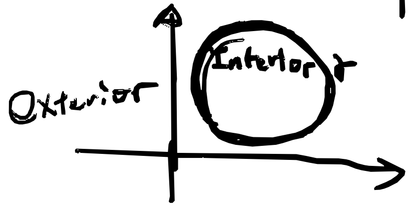
- <u>Jordan curve</u> (simple closed curve) γ

  Let γ be a Jordan Curve in $\mathbb{R}^2$, then it's complement $\mathbb{R}^2 \setminus C$ has exactly 2 connected components. One component is unbounded, called exterior. The other one component is bounded, call interior.

- <u>Rectifiable curve</u>

  A curve with finite length.

Then a rectifiable Jordan curve γ in space G means that γ is finite length and splits G into an interior space and an exterior space.



The trajectory we saw in the Nyquist plot formulates a rectifiable Jordan curve!

We have identified a curve, and we have our transfer function obtained via Laplace transform. This indicates that the transfer function is continuous and continuously differentiable! (analytic)

For those who paid close attention in their complex analysis might recall:

<u>Cauchy's theorem</u>

If G is simply connected space and the curve γ is a rectifiable Jordan curve, then for every analytic function f, we have:

$$\int_\gamma f = 0$$

79 What if there exist a singularity?

Define <u>Index of curve γ</u> (winding number)

$$n(\gamma; a) = \frac{1}{2\pi j} \int_{\gamma} \frac{1}{z-a} dz$$

If γ is a rectifiable Jordan curve in $\mathbb{C}$ and $a \notin \{\gamma\}$.

The result of $n(\gamma; a)$ is always an integer.

When $H(s)$, $s \in \mathbb{C}$ has a zero $s = z$ with multiplicity m. We can write $H(s)$ as

$$H(s) = (s-z)^m G(s) \quad \text{and} \quad G(z) \neq 0$$

We may quickly discover:

$$\frac{H'(s)}{H(s)} = \frac{m}{s-z} + \frac{G'(s)}{G(s)}$$

And $G'(s)/G(s)$ is analytic near $s = z$.

Similarly for a pole $s = p$ with multiplicity n. We can write $H(s) = \frac{G(s)}{(s-p)^n}$, $G(p) \neq 0$. We may find

$$\frac{H'(s)}{H(s)} = \frac{-n}{s-p} + \frac{G'(s)}{G(s)}. \quad \frac{G'(s)}{G(s)} \text{ is also analytic near } s = p.$$

Now we are ready to see the

$\boxed{\text{Cauchy's argument principle}}$

Let $f: G \to \mathbb{C}$ be meromorphic in G with M zeros and N poles counted according to multiplicity. γ is a rectifiable Jordan curve in G that does not go through poles & zeros. Then we have:

$$\frac{1}{2\pi j}\int_\gamma \frac{f'(s)}{f(s)}ds = \sum_{m=1}^{M} n(\gamma; z_m) - \sum_{n=1}^{N} n(\gamma; p_n)$$

Provided the integration over γ is counterclockwise.

On the right side of the equation, we have the number of windings around all zeros minus the number of windings around all poles.

On the left side we have an integral. The geometrical meaning is not immediately obvious. Luckily we may return to the definition of index of curve γ, define the image curve of γ is $\sigma = f\circ\gamma \equiv f(\gamma(t)) : [a,b] \to \mathbb{C}$.

Then we look at the geometrical meaning in the co-domain of f.

[2] Left hand side $\quad (ds = \gamma'(t)\, dt)$

$$\frac{1}{2\pi j}\int_{\gamma} \frac{f'(s)}{f(s)}\, ds = \frac{1}{2\pi j}\int_{a}^{b} \frac{f'(\gamma(t))\,\gamma'(t)}{f(\gamma(t))}\, dt$$

$$= \frac{1}{2\pi j}\int_{a}^{b} \frac{\sigma'(t)}{\sigma(t)}\, dt$$

$$= \frac{1}{2\pi j}\int_{\sigma} \frac{1}{\sigma(t)}\, d\sigma(t)$$

$$= n(\sigma; 0)$$

This reveals the geometrical meaning of the left hand side integral:

the windings of the image curve about the origin.

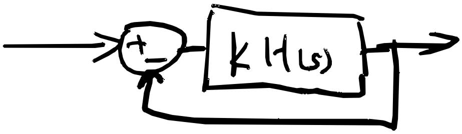
We have

$$n(\sigma; 0) = \sum_{\text{all zeros}} n(\gamma; \text{zero}) - \sum_{\text{all poles}} n(\gamma; \text{poles})$$

Enough of the mathmatical theory.

Back to the transfer function,

in a unity negative feedback control loop:



$$\frac{KH(s)}{1+KH(s)}$$

Recall the stability criteria of a system:

the should not be poles in the open right half plane.

We suppose $H(s) = \dfrac{N(s)}{N(s)}$

then closed-loop transfer function

$$\frac{KH(s)}{1+KH(s)} = \frac{KN(s)}{D(s)+KN(s)}$$

the denominator

$$\boxed{1+KH(s) = \frac{D(s)+KN(s)}{D(s)}}$$

is a "magical" bridge between the open loop process and the closed-loop process.

	Open-loop $H(s)$	$1+kH(s)$	Closed-loop $\frac{kH(s)}{1+KH(s)}$
Simplify	$\frac{N(s)}{D(s)}$	$\frac{D(s)+kN(s)}{D(s)}$	$\frac{KN(s)}{D(s)+KN(s)}$
Poles	$D(s)=0$	$D(s)=0$	$D(s)+KN(s)=0$
Zeros	$N(s)=0$	$D(s)+KN(s)=0$	$KN(s)=0$

① Open-loop poles are $1+kH(s)$ poles

② Closed-loop poles are $1+kH(s)$ zeros

③ Open-loop zeros are closed-loop zeros

Previously we found the following relationship

$$n(\sigma;0) = \sum_{\text{all zeros}} n(\gamma; \text{zero}) - \sum_{\text{all poles}} n(\gamma; \text{poles})$$

In stability analysis, we care about the poles in the right-half plane. In the summary above, we see that we can have knowledge about open-loop & closed-loop poles with $1+KH(s)$. That solve the problem of selecting $f(s) = 1+KH(s)$.

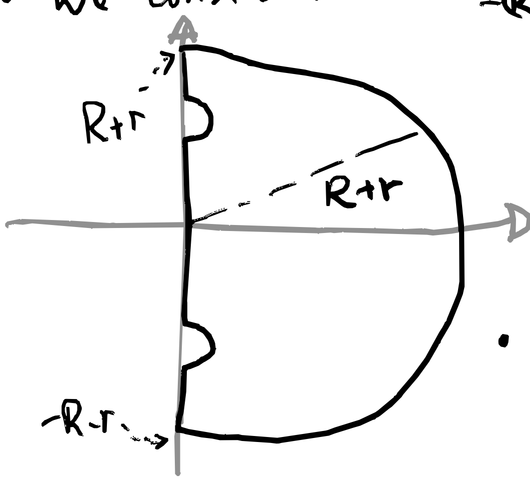
Then we need to select the curve γ.
It would be the most desirable that γ can encircle the entire RHP.
But recall that in all previous theorems γ needs to be rectifiable, and a closed curve containing all RHP is clearily not rectifiable.

We attempt to define a contour Γ of the following:

- we pick an arbitrary small radius r around poles and zeros on the imaginary axis

- we denote largest distance from the origin to a pole or zero as R.

- We construct



- a segment L between $[(R+r)j, -(R+r)j]$ that is mostly on the imaginary axis while skipping the poles & zeros on the imaginary axis with half circle of radius r.
- a half circle Γ_{R+r}: $(R+r)e^{i\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

We have successfully constructed the Nyquist contour that is rectifiable and have all poles and zeros in the open RHP in the interior of the closed contour.

Finally we can safely us the Cauchy's argument principle to formulate the

Nyquist stability criterion:

$$\underbrace{n(\sigma;0)}_{N} = \underbrace{\sum_{\text{all zeros}} n(\gamma;\text{zero})}_{Z} - \underbrace{\sum_{\text{all poles}} n(\gamma;\text{poles})}_{P}$$

$$Z = N + P$$

The closed-loop system is stable if and only if $Z=0$.

Z: # closed-loop transfer function poles

P: # open-loop transfer function poles

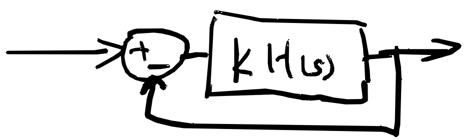
N: # clockwise encirclement at -1 in Nyquist plot of $KH(s)$.

why -1?: we chose $f(s)=1+KH(s)$ but we plot $KH(s)$, so -1.

Ploting Nyquist plot is kind of costly!
Can we determine stability quicker?

We were just looking at the frequency response of the process $H(j\omega)$, can we infer closed-loop stability from Bode plots of $H(j\omega)$?

Yes, we can! (But not always)



$$\frac{H(s)}{1+kH(s)}$$

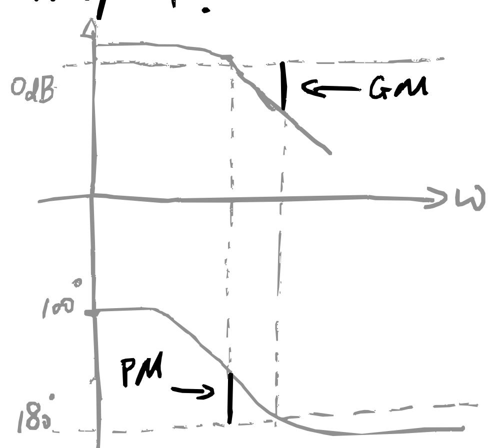
We know that $1+kH(s) \neq 0$, otherwise it will make the close-loop system $\to \infty$.

Thus, if we consider polar form of $kH(s)$

$$|kH(s)|e^{j\theta}$$

Then $\theta \neq \pm 180°$ while $|kH(s)| = 1$.

We look at an example bode plot of a low pass filter. $\longrightarrow$



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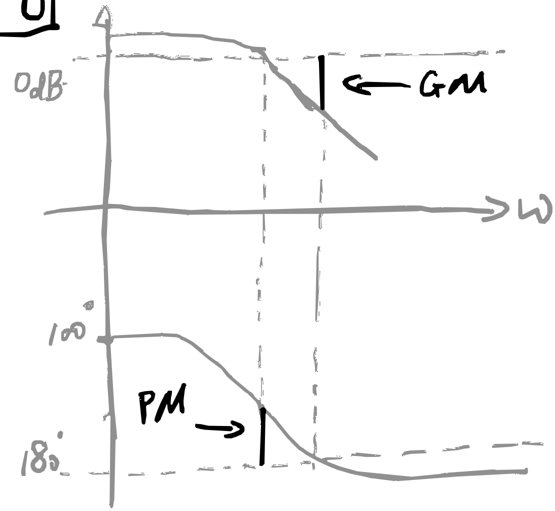


$$|KH| \neq 1$$
$$\theta \neq \pm 180°$$

① when

$$|KH| = 1$$

phase should not be at ±180°

" gain crossover frequency "

ω_{gc}

② when

phase = ±180°

$$|KH| \neq 1$$

the frequency here is called

" phase crossover frequency "

ω_{pc}

Phase margin

$$= \theta_{\omega_{gc}} - 180°$$

or

$$\theta_{\omega_{gc}} - (-180°)$$

Gain Margin

$$= 0dB - |KH(j\omega_{pc})|$$

- Condition for stability in such a case

$$PM > 0 \quad \text{and} \quad GM > 0$$

- Phase margin corresponds to how much delay in time the system can endure before becoming unstable.

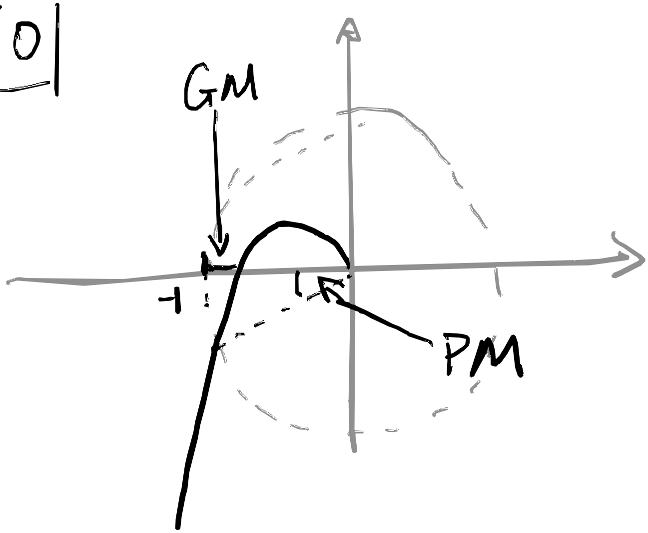
- Gain margin corresponds to how much gain can we increase before the system becomes unstable.

- When phase crosses $\pm 180°$, the sign here:

$$\frac{K\,H(s)}{1 + K\,H(s)}$$

changes! Thus negative feedback becomes positive!

- When gain margin is negative, that means in Nyquist plot we encircle the -1 point, we can not guarantee stability in that case.



What happens if we see in Bode plots there exist multiple gain crossover frequencies and/or phase crossover frequencies?

The analysis depends on the exact system. However, we may always make use of the Nyquist plot and the Nyquist stability criterion.

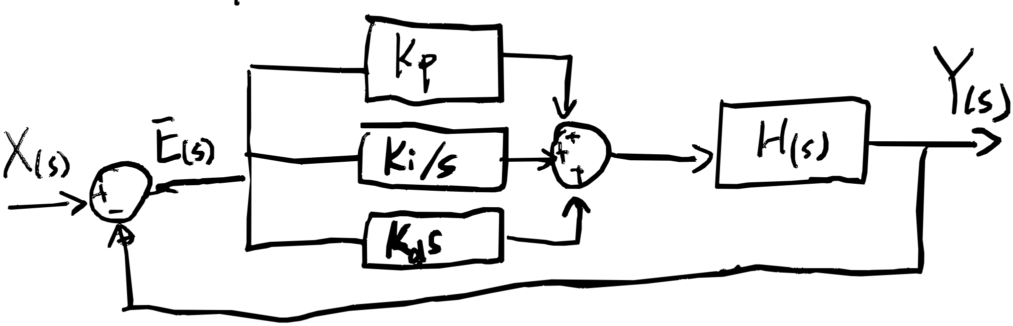
PID Controller

P: proportional controller K_p

I: integral controller K_i/s

D: derivative controller $K_d\,s$

Common implementation: parallel structure



The controller works on the error signal $E(s)$!

Before we dive ourselves into some math, we make an effort to intuitively understand what P, I, D components do.

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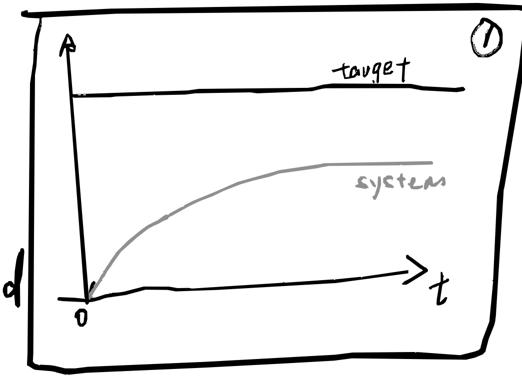
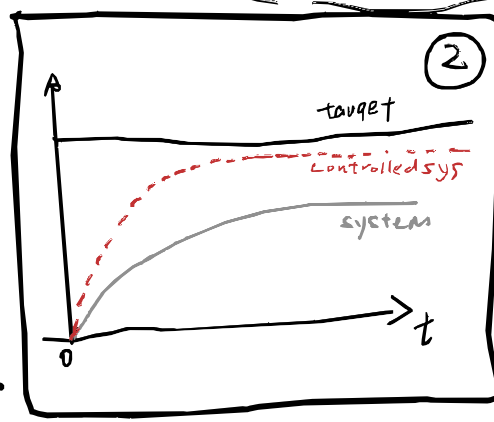


Observe the graph ① on the right, the current system is far from the desired target.



Well, obviously,

- if the response is too small, make it larger by increasing the input.
- if the response is too large, make it smaller by decreasing the input.

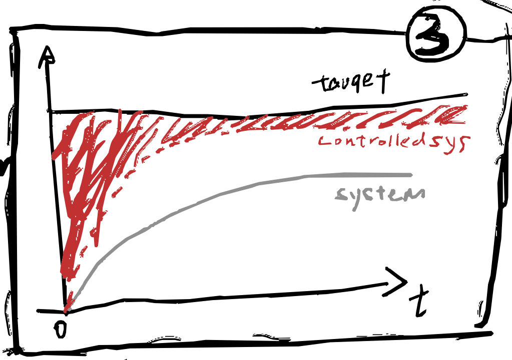
The controller knows the error, so it plays with the error signal by proportionally increase/decrease the gain by a factor K, see graph ②.

But problem can also occur, we have seen in root locus and frequency domain analysis, too large gain could lead to oscillation and further unstable behavior.

Furthermore, if we only use proportional controll, because it only works on the error signal proportionally, it can not fully eliminates the error leading to steady state error.

The shaded area marks the error.

To eliminate the steady state error we can make use of the integral controller.

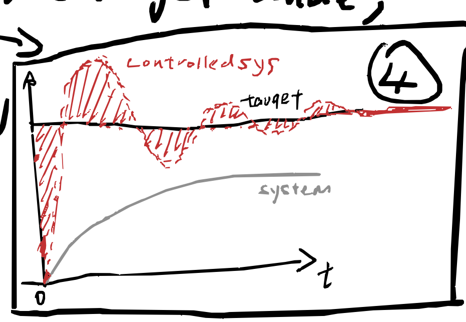


Recall that the controller input is the error, thus when the controller performs the integration, it accumulate all historical error.

Now we can eliminate the steady state error because we can not add/substract instead of only multiply.

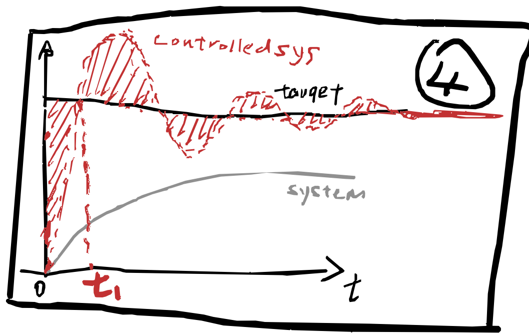
The good side is that fine designed "I" controller will also improve the system's response speed.

But notice that in the situation of 3, the steady state error is small, but does not reach the target, So when an integrator is added, the accumulated historical errors will quickly make the I controller to "exceed" the target value, causing an over-shoot. See 4 → This process will occur recursively until the response rest on the target value in the steady state.



The overshoot induced by the integral controller could be undesirable, the overshoot might exceed system limit. And if you inspect the integration process, you would find that there are redundant efforts to correct the historical accumulated errors, which is less effecient.

But by inspecting the procedure, we can find at t_1, the system already reaches the target value. However, the error keeps accumulating in the integral controller.

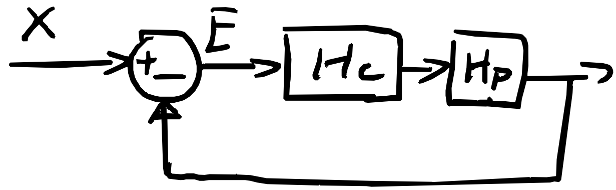


We may find traces of approaching the target value by inspecting the accumulation rate of the cumulative error! As the system approaches the target, the accumulation rate of error becomes slower. Then we can put the derivative information in the controller to eliminate/compensate the excessive effort caused by the integral controller. This is the D - derivative controller.

However, the D controller only deals with change of error, it is sensitive to noise and cannot make the system rest on the steady state target.

In the rest of this section, we derive and prove some of the intuitions mathematically.

- **P-controller**



$$H_c = k_p$$

Steady state error & how large?

Making use of the final value theorem

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{X(s)}{1+H_c(s)H_p(s)}$$

Assume step input $X(s) = \frac{1}{s}$, then

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{1+k_pH_p(s)}.$$

We may decompose $H_p(s) = K_{DC} \dfrac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\cdots}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\cdots}$

thus $\lim_{s\to 0} H_p(s) = K_{DC}$, we have found the stead state error

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{1}{1+k_pK_{DC}}$$

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▲ I - controller

Does it eliminate the steady state error ?

$H_c = K_i/s$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{X(s)}{1+H_c(s)H_p(s)}$$

Assume $X(s) = \frac{1}{s}$ (step input)

$$= \lim_{s\to 0} \frac{s\cdot\frac{1}{s}}{1+\frac{K_i H_p(s)}{s}} = \lim_{s\to 0} \frac{s}{s+K_i H_p(s)} = 0$$

Steady state error $= 0$

▲ D- controller alone

Step response
Steady state error ?

$H_c = K_d s$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{1+K_d s H_p(s)} = 1$$

▲ PD — controller

Step response
Steady state error

$H_c = K_p + K_d s$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \frac{1}{1+K_p H_p(s) + K_d s H_p(s)} = \frac{1}{1+K_p K_{DC}}$$

Ziegler-Nichols Oscillation method for tuning PID controller

① Take only the gain controller $H_c = K_p$

② Increase K_p until the process starts oscillating without damping. (boundary situation)

③ Read the P-controller gain, this gain is called $\underline{K_b}$ - boundary gain.

④ Find the oscillation period $\underline{T}$.

Tune the parameters based on the following table:

	K_p	K_i	K_d
P	$\frac{1}{2}K_b$	—	—
PI	$0.45K_b$	$\frac{1.2}{T}K_b$	—
PID	$0.6K_b$	$\frac{2}{T}K_b$	$\frac{T}{8}K_b$

- Pros:

  Simple, convenient, effective, systematic.

- Cons:

  Driving a system towards boundary of stability & instability is dangerous and costly.

  The resulting closed-loop dynamics can be very different depending on the actual process dynamics.

SUMMARY PID – control

	Advantages	Disadvantages
P	• Fast response	• Potential to be unstable when gain too large • Steady-state error
I	• Eliminate steady-state errors	• Large gain leads to oscillatory behavior
D	• Removes overshoot • Improves stability properties	• Difficult to implement • Sensitive to noise • Magnifies high frequency noise • Steady-state error

Time response characteristics

1st order system

Typical differential equation

$$\frac{dy_{(t)}}{dt} + a y_{(t)} = a x_{(t)}$$

Laplace transform $\quad Y_{(s)}(s+a) = aX_{(s)}$

Transfer function $\quad H_{(s)} = \frac{Y_{(s)}}{X_{(s)}} = \frac{a}{s+a}$

Step Response

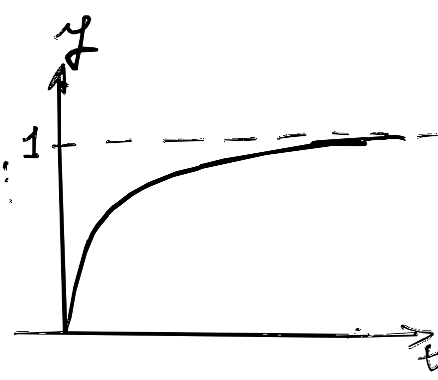
$$X_{(s)} = \frac{1}{s}$$

$$Y_{(s)} = X_{(s)} H_{(s)} = \frac{1}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

We conduct the inverse Laplace transform

$$y_{(t)} = 1 - e^{-at}$$

If $a > 0$, then $y_{(t)}$ looks like:



90 There is a very important concept in 1st order system: $\boxed{\tau}$ — time constant.

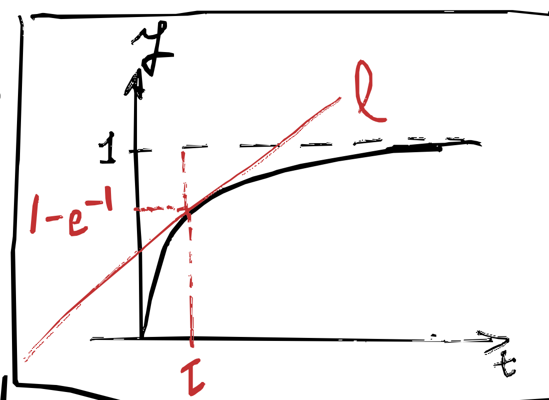
Observe: $y(t) = 1 - e^{-at}$

when we have $t = \frac{1}{a}$,

$$y\left(\frac{1}{a}\right) = 1 - e^{-1} \approx 0.6321$$

We denote $\tau = \frac{1}{a}$

$y'(t) = a e^{-at}$ $\quad y'(\tau) = a e^{-1}$



Assume l: $l(t) = y'(\tau) t + b$

$\Rightarrow 1 - e^{-1} = a e^{-1} \cdot \frac{1}{a} + b$

$\Rightarrow b = 1 - 2e^{-1}$

l: $l(t) = a e^{-1} t + 1 - 2e^{-1}$

we find the intersection between l and the target value 1 by letting $l(t) = 1$

$$a e^{-1} t + 1 - 2e^{-1} = 1 \Rightarrow t = 2\frac{1}{a} = 2\tau$$

Explanation of the calculation:

For a 1st order system

$$H(s) = \frac{1}{\tau s + 1}$$

τ is called — time constant.

- At time τ, the system's step response rise to $1 - e^{-1} \approx 63.21\%$ of the target value.
- If at time τ, the system would fix the rate of change, then it will take another τ time unit to reach the target value.
- If we recursively apply the analysis, we find:

$$y(2\tau) \approx 86.47\%$$

$$y(3\tau) \approx 95.02\%$$

$$y(4\tau) \approx 98.17\%$$

$$\vdots$$

We can then use how many time constants to measure the settling time. For example, if we allow 5% of error then settling time is 3τ, If we allow 2%, then 4τ

9.2

—— Implulse Response ——

$$x(t) = \delta(t) \xrightarrow{\mathcal{L}} X(s) = 1$$

$$Y(s) = X(s)H(s) = \frac{a}{s+a} = \frac{1}{\tau s+1}$$

$$\xrightarrow{\mathcal{L}^{-1}} y(t) = ae^{-at} = \frac{1}{\tau}e^{-t/\tau}$$

Why do we look at impulse response?

if the system has an initial state, say y_0.

The differential equation:

$$\frac{dy(t)}{dt} + ay(t) = ax(t), \quad y(0) = y_0$$

$$\xrightarrow{\mathcal{L}} sY(s) - y_0 + aY(s) = aX(s)$$

$$Y(s) = \underbrace{\frac{a}{s+a}X(s)}_{\text{Step response}} + \underbrace{\frac{y_0}{s+a}}_{\text{Impulse response!}}$$

The time constant also works for step response by just setting 0 as the resting value.

2nd order system

Differential Equation:

$$a\frac{d^2y_{(t)}}{dt^2} + b\frac{dy_{(t)}}{dt} + c\,y_{(t)} = c\,x_{(t)}$$

Laplace transform

$$(as^2+bs+c)\,Y_{(s)} = X_{(s)}$$

Transfer function $\quad H(s) = \dfrac{c}{as^2+bs+c}$

$$= \frac{c}{s^2+\frac{b}{a}s+\frac{c}{a}}$$

let $w_n = \sqrt{\frac{c}{a}}$, $\beta = \dfrac{b}{2\sqrt{ac}}$

we can write transfer function as:

$$H(s) = \frac{w_n^2}{s^2+2w_n\beta s+w_n^2}$$

Characteristic equation:

$$s^2 + 2w_n\beta s + w_n^2 = 0$$

Roots:

$$s = -w_n\beta \pm \sqrt{\beta^2 - 1}\, w_n$$

From previous analysis, we have seen that if the roots contain imaginary number, there will be oscillation. The physical process of removing oscillation is called damping.

Thus we call β: <u>damping factor</u>

when $\beta = 1$, the system is critically damped.

when $\beta > 1$, the system is overdamped.

when $\beta < 1$, the system is underdamped.

when $\beta = 0$, the system is undamped.

Then the transfer function becomes $H(s) \overset{\beta=0}{=\!=} \frac{w_n^2}{s^2 + w_n^2}$

impulse response $Y(s) = X(s)H(s) = w_n \frac{w_n}{s^2 + w_n^2}$

$$\xrightarrow{\mathcal{L}^{-1}} y(t) = w_n \sin w_n t$$

Thus w_n is called the <u>natural frequency</u> of the system.

Step response

$$Y_{(s)} = X_{(s)} H_{(s)} = \frac{\omega_n^2}{s(s^2+2\beta\omega_n s+\omega_n^2)}$$

There are 3 poles:

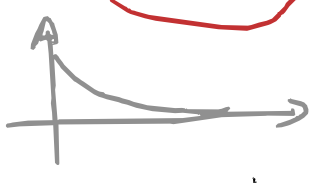
$$p = -\omega_n\beta \pm \sqrt{\beta^2-1}\,\omega_n \ , \ 0$$

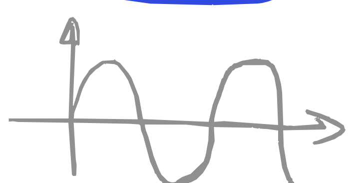
By partial fraction decomposition and inverse Laplace transfer, we find the time domain formula:

$$y_{(t)} = 1 - e^{-\beta\omega_n t}\sqrt{\frac{1}{1-\beta^2}}\ \sin\left(\omega_n\sqrt{1-\beta^2}\,t+\varphi\right)$$

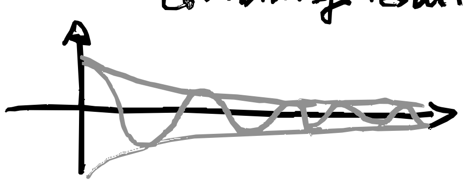
$$\text{with } \varphi = \tan^{-1}\left(\frac{\sqrt{1-\beta^2}}{\beta}\right)$$

$$y_{(t)} = 1 - e^{-\beta\omega_n t}\sqrt{\frac{1}{1-\beta^2}}\ \sin\left(\omega_n\sqrt{1-\beta^2}\,t+\varphi\right)$$



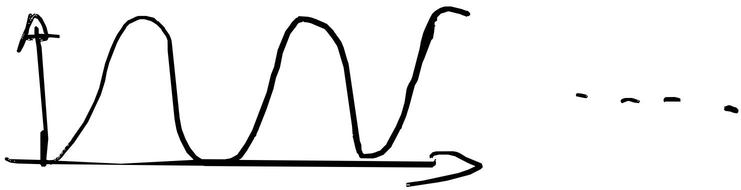


Combining results in



Thus we have:

when $\beta = 0$, $y(t)$ is a sine wave

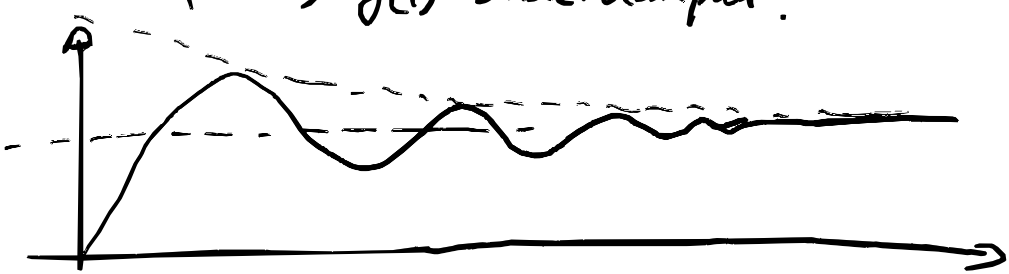


when $\beta \geq 1$, $y(t)$ has no oscillation component

$\beta > 1$: over-damped

$\beta = 1$: critically damped.

when $0 < \beta < 1$, $y(t)$ underdamped.

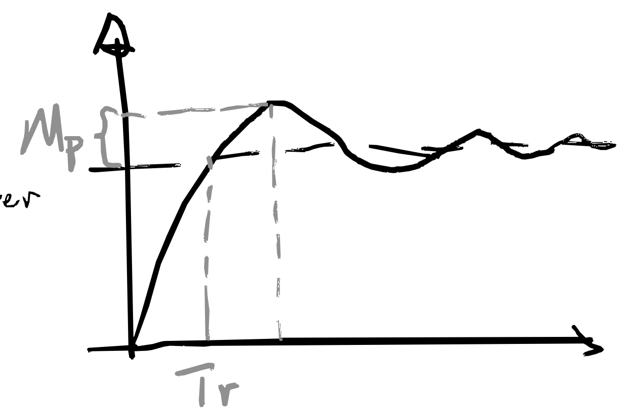


In practical system design, we usually care about:

- Rise time T_r

  time $y(t)$ first reach target value or time $y(t)$ reaches 90% if never reaches the target.

- Maximum Overshoot M_p
- Settling time T_s



Controller design - compensators

Before we have seen analysis tools:

- Root Locus
- Bode plots
- Nyquist plot

In practice, we also care about

- Rise time
- Max overshoot/undershoot
- Settling time

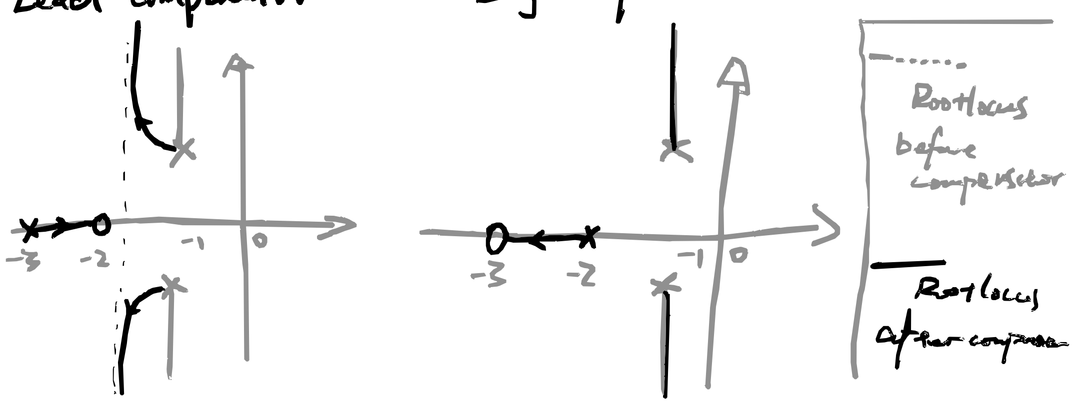
In this chapter we introduce lead compensator and lag compensator, they are of the similar form:

$$\frac{K(s+\frac{w}{\alpha})}{s+w}$$

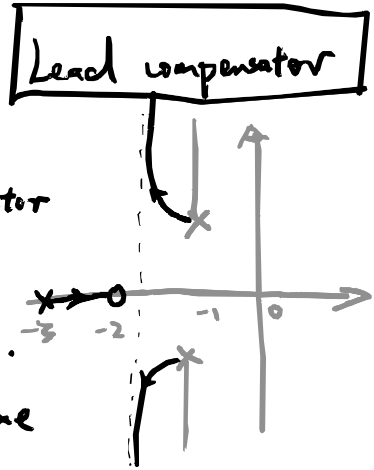
when $\alpha<1$, we achieve a lead compensator.

when $\alpha>1$, we achieve a lag compensator.

Lead compensator   Lag compensator



observe the root locus with and without the lead compensator, we can see that the lead compensator drags the original locus away from the imaginary axis as the gain increase.

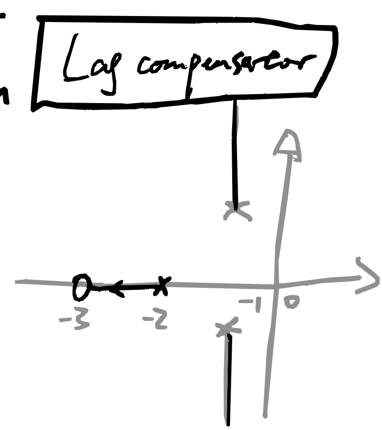


This will improve the settling time of the system, because the "a" on e^{-at} becomes larger, thus the system converge with a steeper curve.

One may argue that a PD-controller can also achieve that. But a PD controller will amplify higher frequency noise.

The reader is encouraged to verify the above statement using previously introduced tools.

Now we look at the lag compensator.



The only change in our example situation is just an additional branch on the real axis.

Why do we do this?

We can reduce steady-state error.

Openloop $H_c(s)H_p(s)$

with $H_c = k_c \frac{(s+\frac{w}{\alpha})}{s+w}$

Closed loop $\frac{1}{1+H_cH_p}$

Final value $\xrightarrow{s \to 0}$ $\frac{1}{1+\alpha k_c K_{DC}}$

without the compensator: $\frac{1}{1+K_{DC}}$

We have successfully reduced the steady state error.

But, PI controller can even remove the steady state error, why do we still need compensators?

The integrator $\frac{1}{s}$ in the PI controller introduces a quarter period (90°) phase drop. This indicates that there is a phase delay which leads to inaccurate tracking. So for our example inputs are simple step responses, if the target value varies with the time, PI controller will be less desirable.

Lag compensators can also improve phase margin.

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Block diagrams

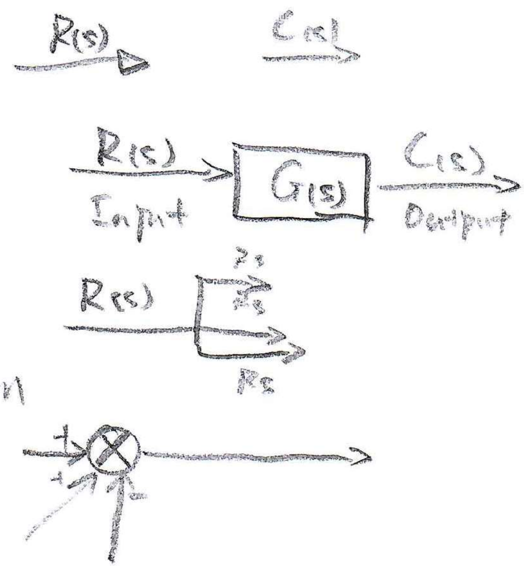
[redacted]

why using it ?

- Blocks draw easier than real physical systems.
- Blocks all look the same. Systems look alike (analogy)
- Block diagrams are easier to read.
- Easier to manipulate & calculate — Block properties / Rules.

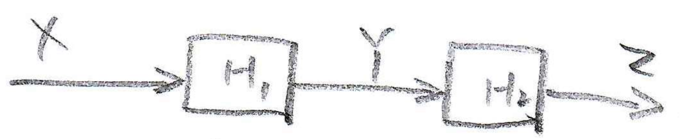
ELEMENTS.

- Signals
- Blocks (systems).
- Pickoff point
- Summing Junction



BLOCK Properties

1. Series.



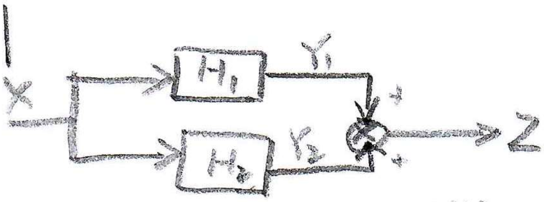
$Y = H_1 \cdot X$ & $Z = H_2 \cdot Y$

HENCE $Z = H_1 \cdot H_2 \cdot X$.

new

$H_{new} = H_1 \cdot H_2$; $Z = H_{new} \cdot X$

2. Parallel

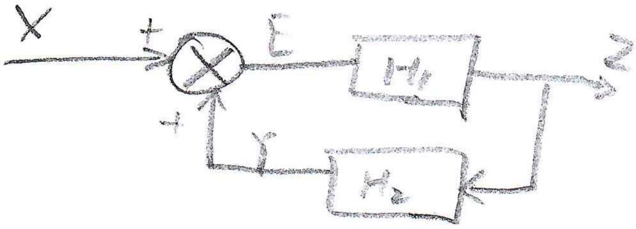


$Y_1 = H_1 \cdot X$; $Y_2 = H_2 \cdot X$. new → $H_{new} = H_1 + H_2$

$Z = Y_1 + Y_2 = (H_1 + H_2) X$

$Z = H_{new} \cdot X$

3. Positive feedback.



$$Z = H_1X + H_1H_2Z$$

$$(1-H_1H_2)Z = H_1X$$

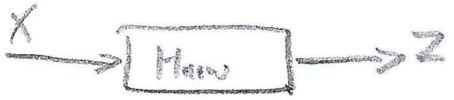
$E = X+Y$, $Y = H_2 \cdot Z$

HENCE $E = X + H_2 \cdot Z$

$Z = H_1 \cdot E$

HENCE $Z = H_1 \cdot (X + H_2 Z) \Longleftrightarrow Z = \dfrac{H_1}{1-H_1H_2} X$

$$H_{new} = \frac{H_1}{1-H_1H_2}$$



4. Negetive feedback.

$E = X - Y$,

$Y = H_2 \cdot Z$

$\Rightarrow E = X - H_2 \cdot Z$

$Z = H_1 \cdot E$

$\therefore Z = H_1 \cdot (X - H_2 Z)$

$\Rightarrow Z = \dfrac{H_1}{1+H_1 H_2} X$

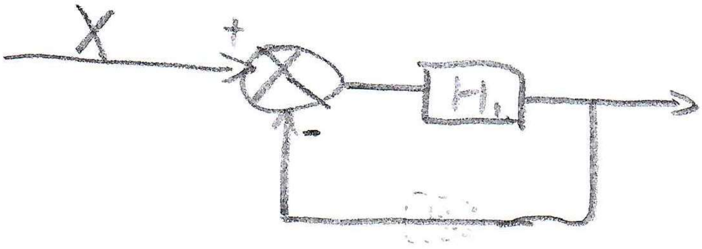
$$H_{new} = \frac{H}{1+H_1H_2}$$

ALTERNATIVE WAY.

H_{new}

$$= \frac{H_{forward}}{1 + H_{loop}}$$

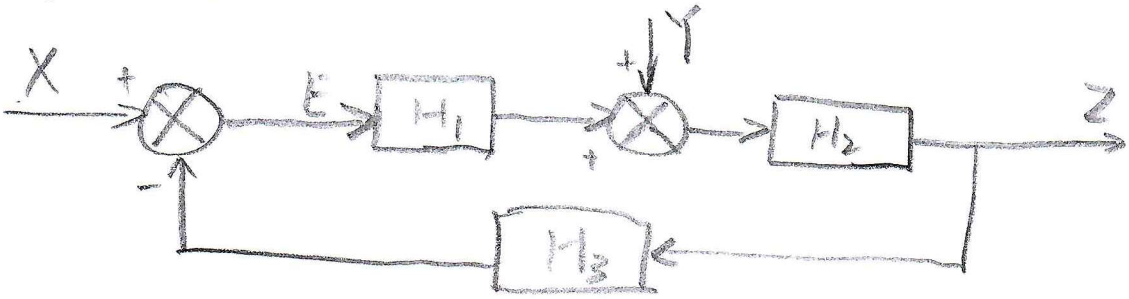
5. Unity feedback.



$H_2 = 1$.

So, $H_{new} = \dfrac{H_1}{1+H_1}$

6. Disturbance.



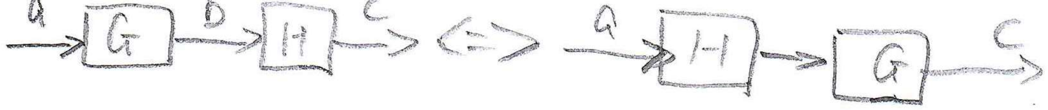
$E = X - H_3 \cdot Z$ & $Z = H_1 \cdot H_2 \cdot E + H_2 \cdot Y$.

$$\Rightarrow Z = \underbrace{\frac{H_1 \cdot H_2}{1+H_1 \cdot H_2 \cdot H_3}}_{\downarrow} X + \underbrace{\frac{H_2}{1+H_1 \cdot H_2 \cdot H_3}}_{\downarrow} Y.$$

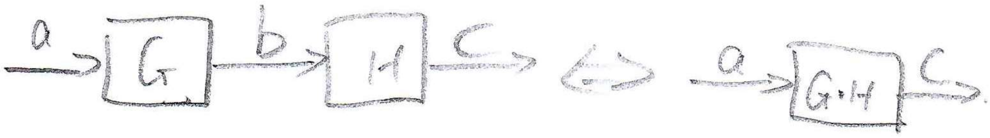
$$Z = H_{control} \cdot X + H_{disturbance} \cdot Y.$$

More Rules to Modify Block diagrams.

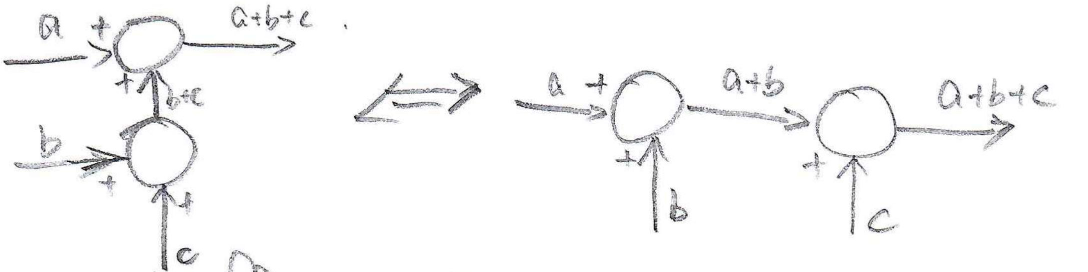
- Exchanging elements.



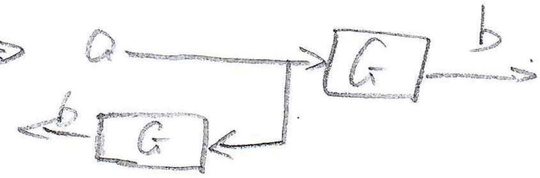
- Combining elements



- Regroup summing junction



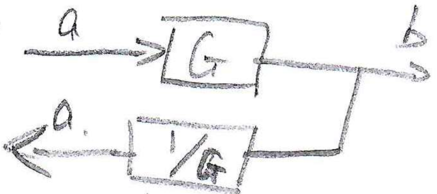
- Move pick-off point ①



$b = Ga$

- Move pick-off point ②

⟺

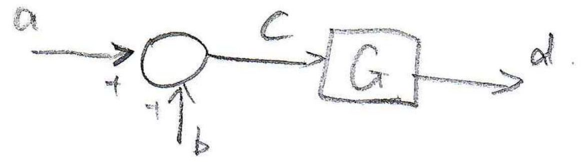


$aG = b$.

$b \cdot \frac{1}{G} = a$

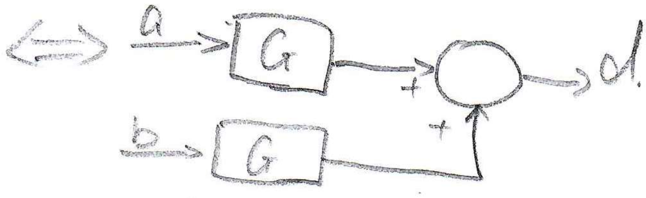
- Move summing junction ①.

- Move summing junction ②



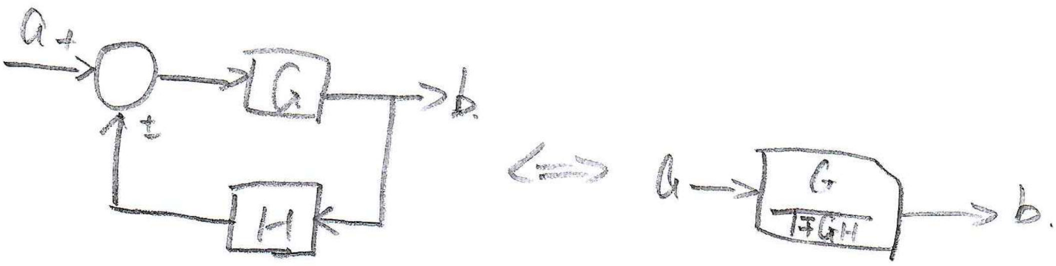
$c = a+b$

$d = (a+b)G = aG + bG$

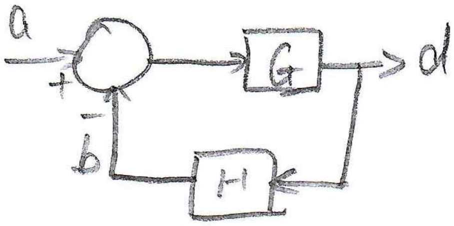


$aG + bG = d$

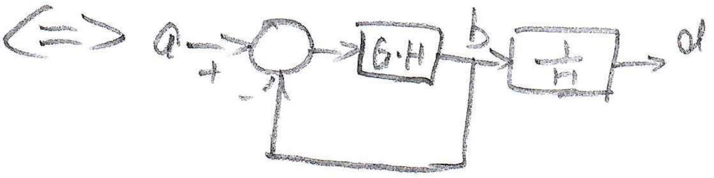
- Eliminate feedback loop.



- Remove element from feedback loop.



$d = \frac{G}{1+G \cdot H} a$

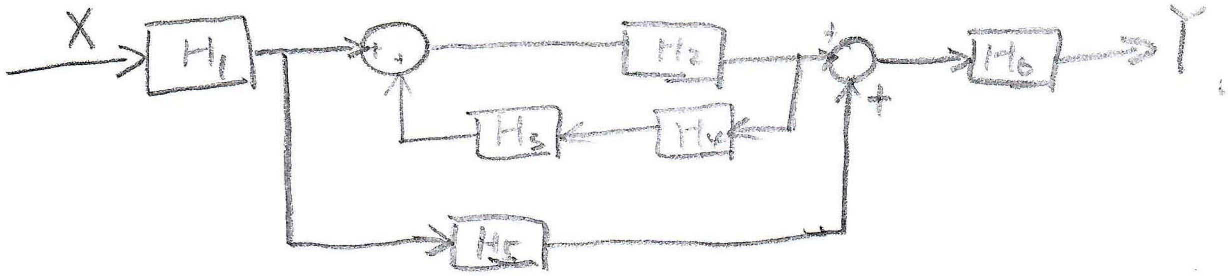


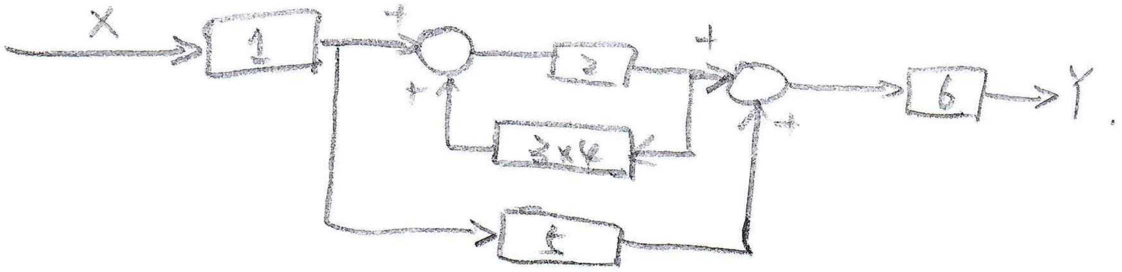
$b = \frac{GH}{1+GH} \cdot a$

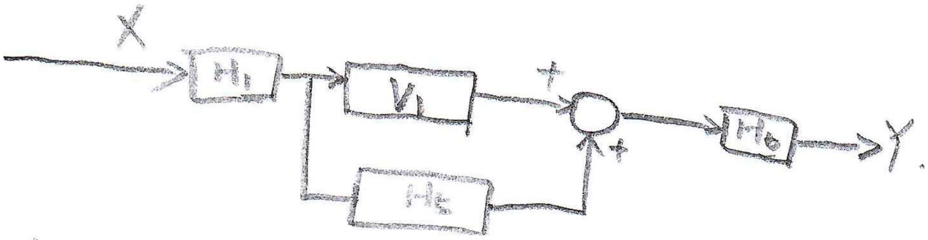
$d = \frac{b}{H} = \frac{G}{1+GH} \cdot a$

- Combining Elements

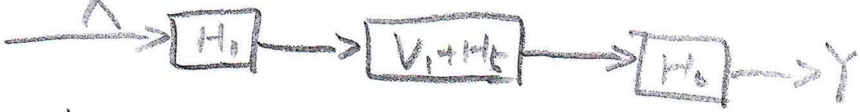
Block diagram Excerise 1.

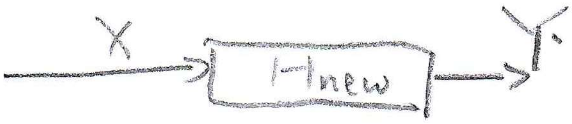






$$V_1 = \frac{H_2}{1 - H_2H_3H_4}$$

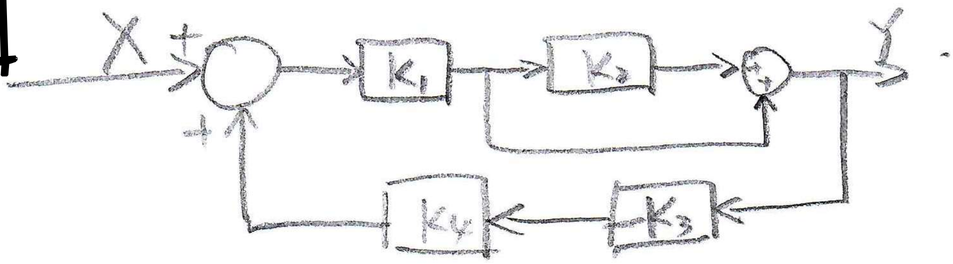


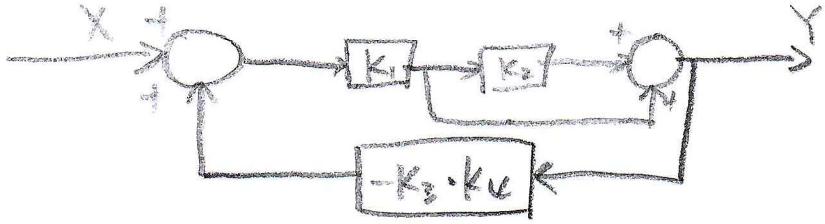


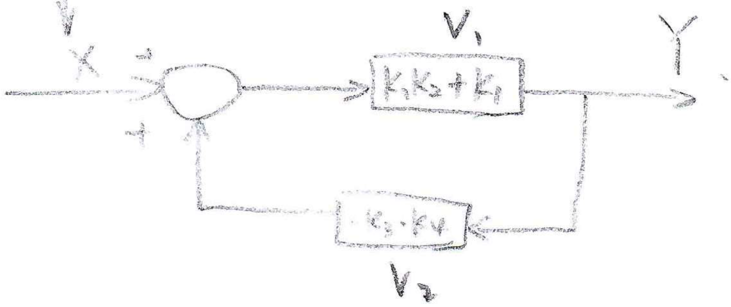
$$H_{new} = H_1 \cdot \left(\frac{H_2}{1-H_2H_3H_4} + H_5\right) \cdot H_6$$

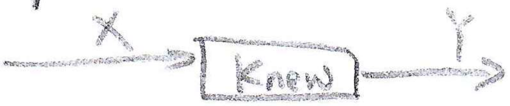
$$= \frac{H_6 \cdot H_1 \cdot H_2}{1-H_2H_3H_4} + H_6 \cdot H_1 \cdot H_5$$

BLOCK diagram Excerise 2.



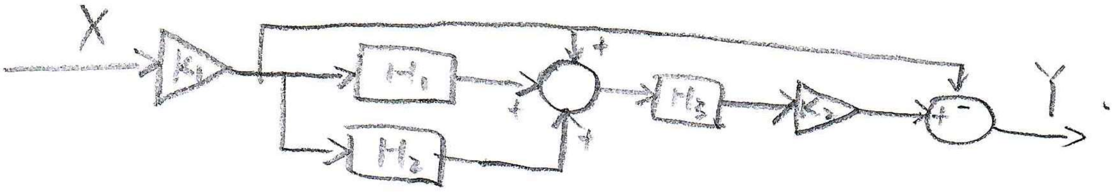


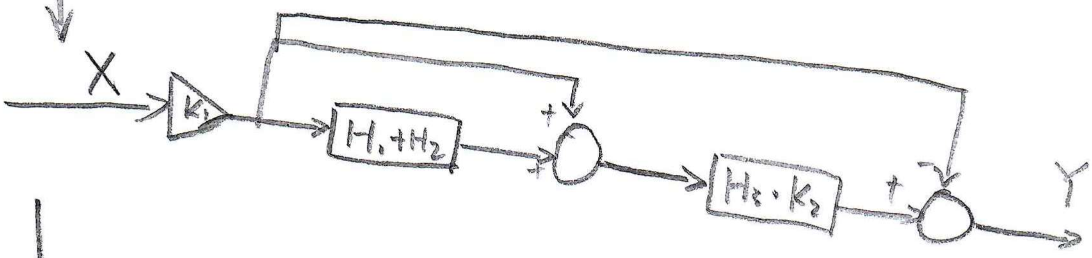


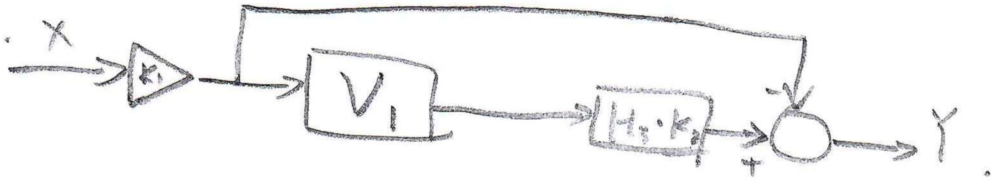


$$K_{new} = \frac{V_1}{1 - V_1 V_2} = \frac{K_1 + K_1 K_2}{1 + K_1 K_2 K_3 K_4 + K_1 K_3 K_4}$$

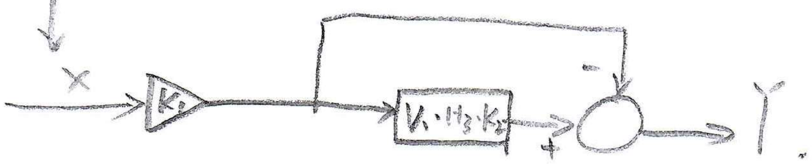
Block diagram Excercise 5.

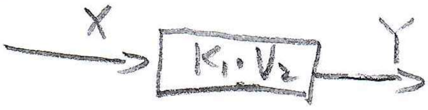






$V_1 = 1 + H_1 + H_2$.





$V_2 = V_1 \cdot H_3 \cdot K_2 - 1$.

$\Longrightarrow$ $H_{new} = K_1 \cdot V_2$.

$= K_1 \cdot [(1+H_1+H_2) \cdot H_3 \cdot K_2 - 1]$

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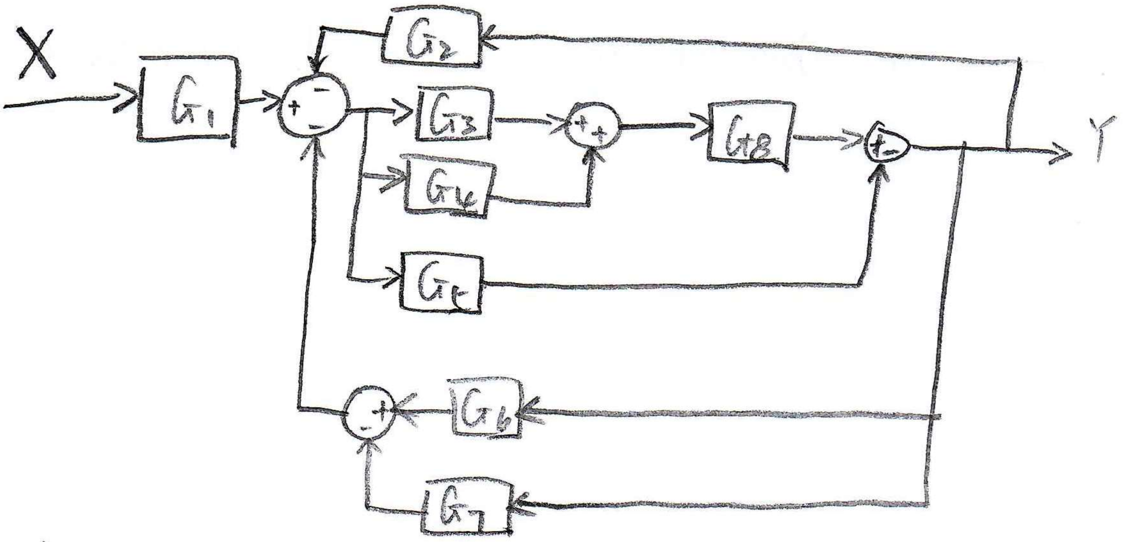
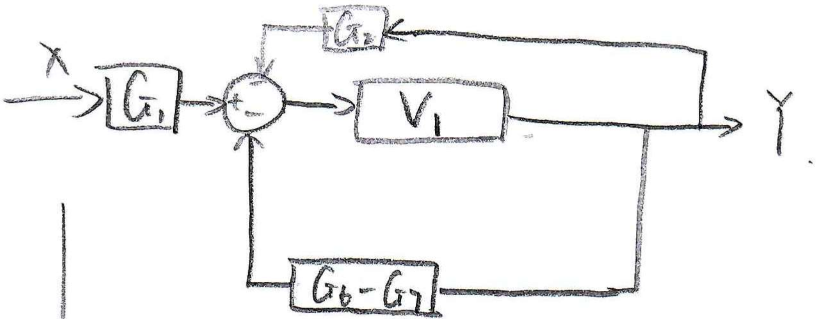
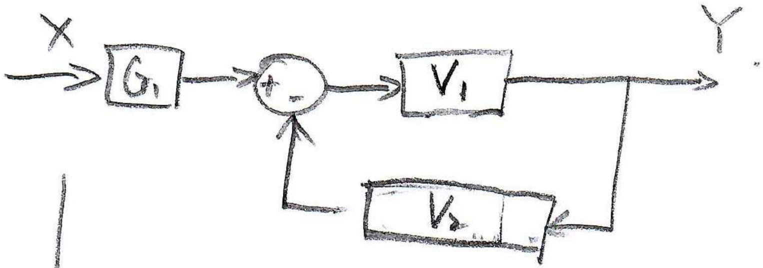
Block diagram Excerise 4.

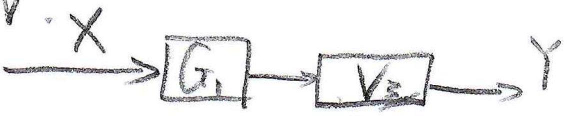




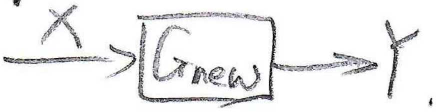
$$V_1 = (G_3+G_4)\cdot G_8 - G_5$$



$$V_2 = G_6 - G_7 + G_2.$$

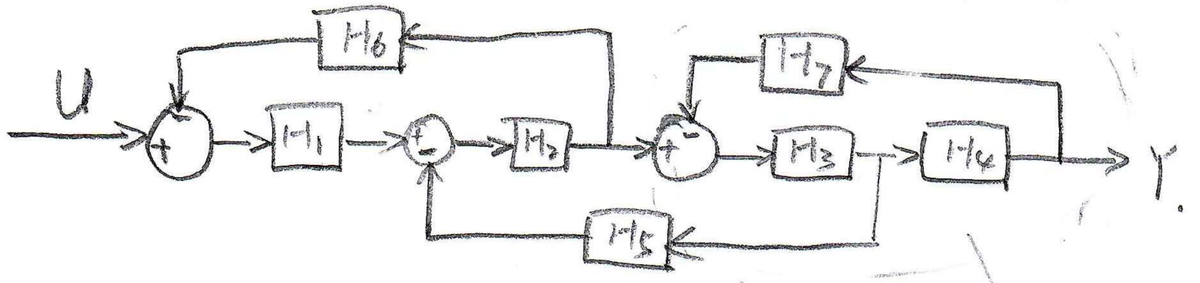


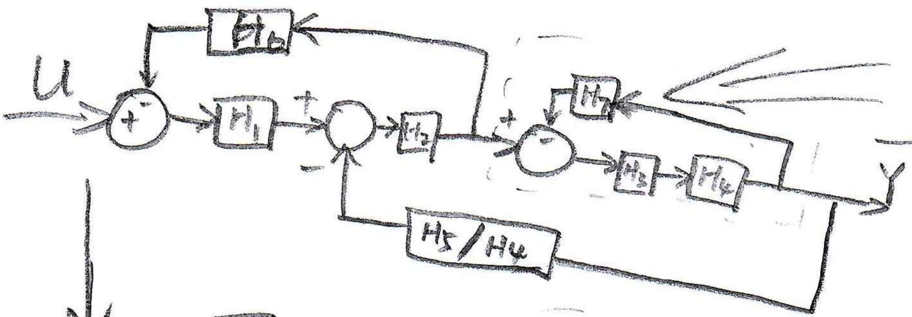
$$V_3 = \frac{V_1}{1+V_1\cdot V_2}$$



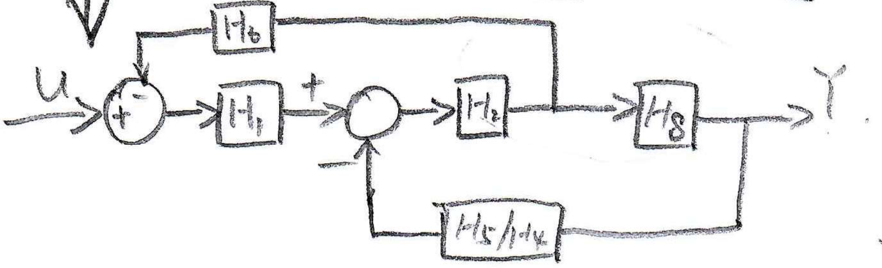
$$G_{new} = V_3\cdot G_1 = G_1\cdot\frac{(G_3+G_4)G_8 - G_5}{1+[(G_3+G_4)G_8 - G_5]\cdot[G_6-G_7+G_2]}$$

AN EXAMPLE — On how to deal with mixed loops

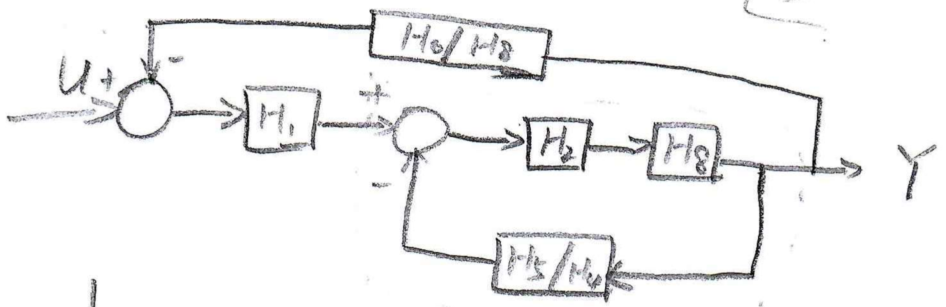


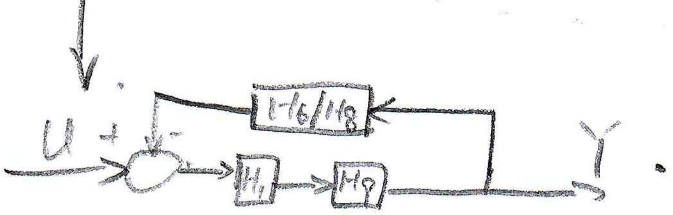


the ideal thing is to create a seperate loop to solve this part.

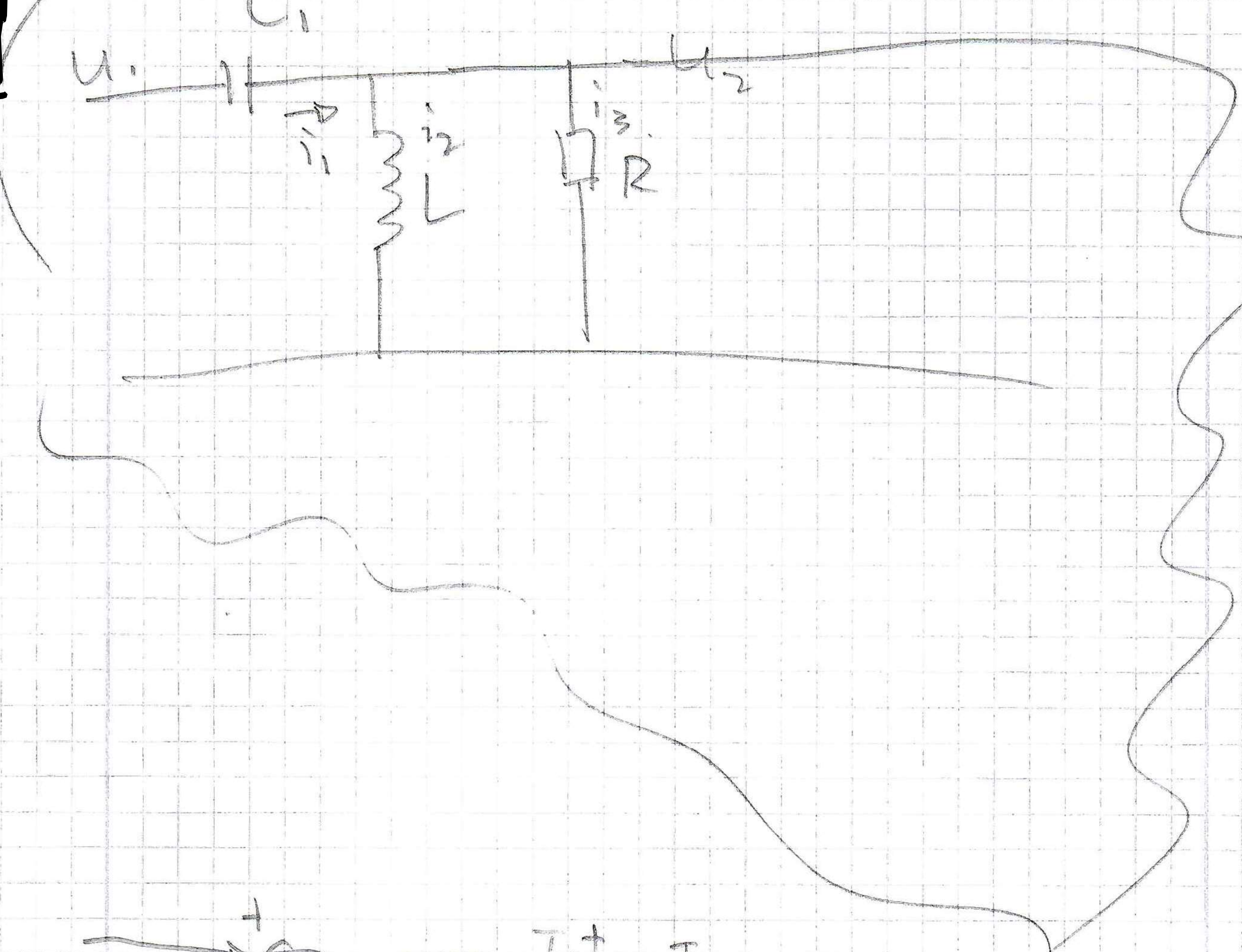


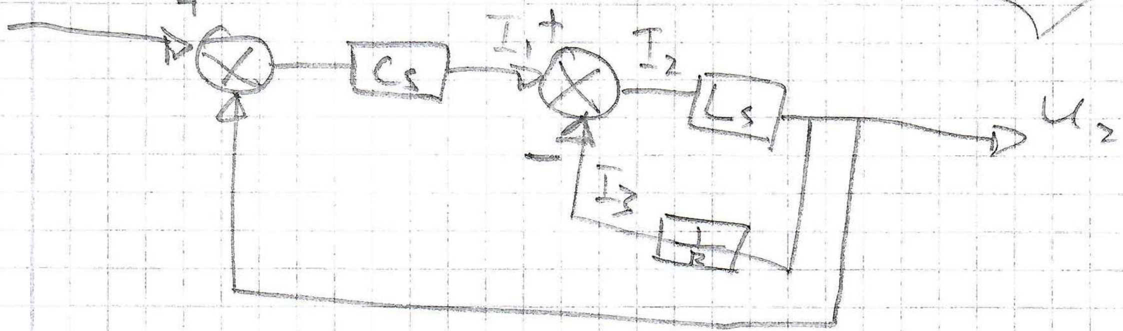
Another mixed loop.





110 FORMULATING BLOCK DIAGRAM





$$i_1 = i_2 + i_3$$

$$i_1 = C \cdot \frac{du_{(t)}}{dt}$$

$$u_2 = L \frac{di_{(t)}}{dt} \qquad i_2 = \frac{U_{2(s)}}{Ls}$$

$$i_3 = \frac{u_2}{R}$$